A NOTE ON THE RECONSTRUCTION OF SETS OF FINITE MEASURE

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Abstract. We prove a measure-theoretic version of a result due to Radcliffe and Scott [8] from 1999 about the reconstruction of infinite sets of real numbers. This answers their question from [8].

1. Introduction

Combinatorial reconstruction problems have their roots in two well-known open conjectures about finite graphs, the Reconstruction Conjecture of Kelly [5], [6] and Ulam [12] and the Edge Reconstruction Conjecture of Harary [4] (cf. Bondy’s survey [3]). A typical reconstruction result states that some combinatorial object is uniquely determined up to some notion of isomorphism by the collection of its sub-objects of a fixed size given up to isomorphism. This collection is usually called the deck of the considered object.

In [8] Radcliffe and Scott defined the $k$-deck $d_{A,k}$ of some set $A \subseteq \mathbb{Z}$ of integers and for some positive integer $k \in \mathbb{N}$ as a function defined for all sets $S \subseteq \mathbb{Z}$ of at most $k$ integers by $d_{A,k}(S) = \left| \{ i \in \mathbb{Z} \mid S + i = \{ s + i \mid s \in S \} \subseteq A \} \right|$. It is very easy to see that every finite set $A \subseteq \mathbb{Z}$ is uniquely determined up to translation by its $3$-deck (cf. [8]). Still in [8], Radcliffe and Scott considered infinite sets $A \subseteq \mathbb{Z}$ such that the $2$-deck $d_{A,2}$ — and hence all $k$-decks for all $k \geq 2$ — takes only finite values. They called such sets locally finite and proved that every locally finite set is uniquely determined up to translation by its $3$-deck. We generalized their result in [9] showing that for every $k \in \mathbb{N}$ with $k \geq 3$ every set $A \subseteq \mathbb{Z}$ such that $d_{A,k}$ takes only finite values is uniquely determined up to translation by $d_{A,k}$ and one finite non-zero value of $d_{A,k-1}$. Similar problems have been considered in [10] and [11].

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Radcliffe and Scott close [8] with the question whether a measure-
theoretic version of their result holds. We answer their question by proving
such a version in the present paper.

Let $A \subseteq \mathbb{R}$ be a (Lebesgue-)measurable set of finite measure

$$\mu(A) = \int_{\mathbb{R}} 1_A(u) \, du < \infty$$

where $1_A$ denotes the characteristic function of $A$. For simplicity, we will
abbreviate $\int_{\mathbb{R}}$ as $\int$.

For $k \in \mathbb{N}$ and $x_1, x_2, \ldots, x_k \in \mathbb{R}$ (not necessarily distinct) the $k$-deck
d$A_k$ of $A$ on the set $X = \{x_1, x_2, \ldots, x_k\}$ with $|X| \leq k$ is defined by

$$d_{A,k}(X) = \int \prod_{j=1}^{k} 1_{A}(u + x_j) \, du.$$ 

Since $1_A \in L^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \int |f(u)| \, du < \infty\}$, the $k$-deck $d_{A,k}(X)$
can be considered as the $k$-fold convolution of functions in $L^1(\mathbb{R})$ and hence
it is itself in $L^1(\mathbb{R}^k)$ considered as a function of $(x_1, x_2, \ldots, x_k)$ (cf. [7]).
Trivially, $d_{A,1}(x) = \mu(A)$ for all $x \in \mathbb{R}$.

Our approach uses some Fourier analysis and for definitions and basic
theorems we refer the reader to [1] or [2]. For some function $f \in L^1(\mathbb{R})$, the
Fourier transform $\hat{f}$ is the function in

$$\text{C}_0(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } \lim_{|u| \to \infty} f(u) = 0 \right\}$$

defined for $v \in \mathbb{R}$ by $\hat{f}(v) = \int f(u)e^{-iuv} \, du$.

Our first lemma relates the $k$-deck of some set $A \subseteq \mathbb{R}$ and the Fourier
transform of its characteristic function $1_A$.

**Lemma 1.1.** Let $A \subseteq \mathbb{R}$ and $k \in \mathbb{N}$ be such that $1_A \in L^1(\mathbb{R})$. Then for
all $x_1, x_2, \ldots, x_k \in \mathbb{R}$

$$\tilde{1}_A \left( \sum_{j=1}^k x_j \right) \cdot \prod_{j=1}^k \tilde{1}_A(x_j) = \int \cdots \int d_{A,k+1}(0, w_1, w_2, \ldots, w_k)$$

$$\cdot \prod_{j=1}^k e^{i x_j w_j} \, dw_1 \, dw_2 \cdots dw_k.$$