EXPONENTIALLY LOCALIZED SOLUTIONS OF THE KLEIN–GORDON EQUATION

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UDC 550

Exponentially localized solutions of the Klein–Gordon equation for two and three space variables are presented. The solutions depend on four free parameters. For some relations between the parameters, the solutions describe wave packets filled with oscillations whose amplitudes decrease in the Gaussian way with distance from a point running with group velocity along a ray. The solutions are constructed by using exact complex solutions of the eikonal equation and may be regarded as ray solutions with amplitudes involving one term. It is also shown that the multidimensional nonlinear Klein–Gordon equation can be reduced to an ordinary differential equation with respect to the complex eikonal. Bibliography: 12 titles.

Dedicated to Vasili Mikhailovich Babich
on the occasion of his anniversary

Introduction. The construction of various highly localized solutions of the wave equation

$$\phi_{tt} - \phi_{xx} - \phi_{yy} - \phi_{zz} = 0$$

is a subject matter of several publications (see [1–8]). In particular, in [1] a solution that is exponentially localized in the vicinity of a point running with velocity of light is given. In this paper, we generalize the result of [6] by presenting a family of localized solutions that includes the one given in [6] as a special case.

Based on the results obtained for the wave equation, we construct a family of particle-like solutions of the Klein–Gordon equation

$$h^2 (u_{tt} - u_{xx} - u_{yy}) + u = 0, \quad h = \text{const.}$$

These solutions have finite energy and describe wave packets with central frequency $\omega$ and wave number $k$, where $\omega^2 = k^2 + 1/h^2$. Their amplitudes decrease exponentially with distance from a point running along a straight line with group velocity $v = d\omega/dk$. By analogy with the solutions of the wave equation, we call them Gaussian wave packets.

All the solutions of the Klein–Gordon equation from this class can be represented in the form

$$u = A f(iS/h),$$

where the function $S$ satisfies the eikonal (or Hamilton–Jacobi) equation

$$S_t^2 - S_x^2 - S_y^2 = 1,$$

the amplitude factor $A$ does not depend on the coordinates, $S$ and $A$ are independent on $h$, and $f$ is expressed in terms of the Hankel function.

The comparison of our results with those available in the current literature shows that one of the solutions that we found for the three-dimensional Klein–Gordon equation coincides with the solution given in [7]. We also consider in detail another solution from the family constructed, which depends only on one variable $S$ that is one of the exact complex solutions of the eikonal equation. It turned out that the search for a solution dependent only on $S$ of the nonlinear Klein–Gordon equation is reduced to the solution of an ordinary differential equation. This is also true for nonlinear Klein–Gordon equations with arbitrary number of space variables.

In constructing particle-like solutions of the wave equation, we followed the idea by Ziolkowski [3] and sought such solutions in the form of a superposition of Gaussian beams, which are solutions localized near rays. The latter solutions were found for the first time in papers of Brittingham [1] and Kiselev [2] and belong to the class of relatively undistorted waves, in the terminology of Courant and Hilbert [10].

The construction of the solution of the Klein–Gordon equation employs the simple observation that, taking the Fourier transform of a solution of wave equation (1), say with respect to $z$,

$$u(x, y, t) = \int_{-\infty}^{\infty} dz \, \phi(x, y, z, t) \, e^{i z / h},$$

we obtain a solution of the Klein–Gordon equation (2).
Generalized Gaussian packets for the wave equation. We start from the solution of wave equation (1), described by the formula
\[ \phi_b(x, y, z, t, q) = \exp \left\{ i q \Theta_1 \right\} \left( \beta - i \varepsilon_1 \right)^{1/2} / \left( \beta - i \varepsilon_2 \right)^{1/2} \] (5)
(see [1, 2, 6]), where the notation
\[ \Theta_1 = x - t + \frac{y^2}{\beta - i \varepsilon_1} + \frac{z^2}{\beta - i \varepsilon_2}, \quad \beta = x + t, \] (6)
is introduced. The function \( \phi_b \) satisfies (1) for any \( q, \varepsilon_1, \) and \( \varepsilon_2, \) and in the case of \( \varepsilon_1 > 0, \varepsilon_2 > 0, \) and \( q > 0 \) it is a Gaussian beam, which means that it is localized in the Gaussian way in the vicinity of the \( x \) axis.

We seek particle-like solutions of Eq. (1) in the form of a superposition of Gaussian beams:
\[ \phi^{(\nu)}_p(x, y, z, t) = \int_0^\infty dq \, F^{(\nu)}(q) \phi_b(x, y, z, t, q), \] (7)
where \( F^{(\nu)}(q) \) is a specialized function dependent on the parameter \( \nu. \) We put
\[ F^{(\nu)}(q) = a \, q^{-\nu - 1} e^{-\nu (q + \sigma^2)/q}, \] (8)
where \( \nu, \sigma, \) and \( \varepsilon \) are arbitrary constants, \( \sigma > 0, \varepsilon > 0, \) and \( a = (4 \varepsilon \sigma^2)^{\nu}/(2 \sqrt{\pi}) \). It can easily be shown that (7) is reduced to an integral representation of the Hankel function \( H^{(1)}_\nu \) of the first kind [12] and
\[ \phi^{(\nu)}_p(x, y, z, t) = C s^{2 \nu} H^{(1)}_\nu(s), \quad s = 2 i \sigma \varepsilon \left( 1 - 2 \Theta_1 / \varepsilon \right)^{1/2}, \] (9)
where \( C = i \, 2^{\nu - 1} \sqrt{\pi}. \) It is worth noting that \( s \) satisfies the Hamilton–Jacobi equation \( s^2 = s_x^2 + s_y^2 + s_z^2 \) for wave equation (1).

We note that for \( \nu = 1/2, \) formula (9) yields a solution of the wave equation presented earlier in [6]:
\[ \phi^{(1/2)}_p(x, y, z, t) = \exp \left\{ -2 \sigma \varepsilon \sqrt{1 - 2 \Theta_1 / \varepsilon} \right\} \left( \beta - i \varepsilon_1 \right)^{1/2} / \left( \beta - i \varepsilon_2 \right)^{1/2}. \] (10)
This solution depends on four free parameters \( \varepsilon, \varepsilon_1, \varepsilon_2, \) and \( \sigma. \) It is established in [6] that if all these parameters are positive, it is localized in the Gaussian way near the point \( x = y = 0 \) and \( z = ct \) that runs with velocity of light \( c = 1 \) along the \( x \) axis.

The asymptotics of the solutions of (1) of the form (9) with respect to the large argument have the same exponential factor as (10). Arguments similar to those adduced in [6] prove their localization. Therefore, (9) is a localized solution generalizing (10).

Gaussian beams for the Klein–Gordon equation. Here we give a solution \( u_b \) of the Klein–Gordon equation, which has a Gaussian localization near a ray, in order to use it in the sequel in the construction of particle-like solutions of the Klein–Gordon equation.

To find such a \( u_b, \) we calculate the Fourier transform (4) with respect to \( z \) of expression (5):
\[ u_b(x, y, t, q) = \frac{\sqrt{\pi} e^{i \pi/4 - \varepsilon_2/(4 q^2)}}{\sqrt{q}} \exp \left\{ i \Theta q - i \beta/(4 q^2) \right\} / \sqrt{\beta - i \varepsilon_1}, \] (11)
where
\[ \Theta = x - t + \frac{y^2}{\beta - i \varepsilon_1}. \] (12)

This solution was found first in [8]. We call such solutions of the Klein–Gordon equation Gaussian beams, by analogy with solutions of wave equations.