A WAVE-FRONT CURVATURE APPROACH TO COMPUTING RAY AMPLITUDES IN INHOMOGENEOUS MEDIA WITH CURVED INTERFACES

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Summary: A system of three ordinary non-linear first order differential equations is proposed for the computation of the geometrical spreading of the wave front of a seismic body wave in a three-dimensional medium. The variables of the system are the parameters which provide a second order approximation of the wave front.

1. INTRODUCTION

A novel approach to computing the function $J$ has been suggested by Popov and Plenčík [4, 5]. It is based on choosing a co-ordinate system which moves with the wave front along the ray in a well-defined manner. The method demands the integration of a so-called additional system of 3 first order linear differential equations along a ray $r(s)$ that has been previously established by some arbitrary ray tracing system. $s$ is the arc length measured along the ray from some initial ray point $s = s_0$. The velocity of propagation of elastic waves is described by the function $c = c(x, y, z)$ which is expected to be continuous together with its spatial derivatives up to the second order. A significant role of the theory is attributed to the particular choice of co-ordinates $(s, q_1, q_2)$ that pertain to the specified ray. Equally important is the introduction of the angle

$$\begin{align*}
\Theta(s) &= \Theta(s_0) + \int_{s_0}^{s} T(t) \, ds,
\end{align*}$$

where $T(s)$ is the ray torsion.

At each ray point $s$ let a plane $S_1$ be defined which is perpendicular to the ray. One may then specify two mutually perpendicular unit vectors $e_1$ and $e_2$ (with their origins on the ray) by means of the following relations

$$\begin{align*}
e_1 &= n \cos \Theta - b \sin \Theta, \quad e_2 = n \sin \Theta + b \cos \Theta.
\end{align*}$$

Here $n$ and $b$ are the unit normal and binormal vectors of the ray in $r(s)$. The construction of $e_1$ and $e_2$ only requires parameters that can be derived from the ray. The position of some arbitrary point $M$ on $S_1$ can be expressed by the location vector

$$\begin{align*}
r_M &= r(s) + q_1(s) \, e_1(s) + q_2(s) \, e_2(s).
\end{align*}$$

$r_M$ is expected to be close to $r(s)$. The unit vectors $e_1, e_2, t$ (t being the unit vector tangent to the ray, pointing along the direction of the propagating wave front) form a right-hand system of unit vectors.

It is assumed that the reader of this paper is familiar with the publications [4, 5] as the method developed here is largely based on results obtained by Popov and Plenčík. Please also note that

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our notation is identical with the one used in [4, 5] unless otherwise stated. The additional system of 8 equations that can be used to determine $J$ and which is derived in [4, 5] reads as follows

\begin{equation}
(4a, b) \quad \frac{dQ_{ji}}{ds} = cP_{ji}, \quad \frac{dP_{ji}}{ds} = -c^{-2}(c_{ji}Q_{ii} + c_{j2}Q_{21}).
\end{equation}

The quantities $Q_{ji}$, $P_{ji}$ and $c_{ji}$ ($i, j = 1, 2$) have the following meaning: $Q_{ji} = \partial q_j/\partial x_i$, $P_{ji} = \partial p_j/\partial x_i$, $c_{ji} = \partial c/\partial q_i \partial q_j$, $a_1$ and $a_2$ are the ray parameters. All quantities in (4) are considered for $q_1 = q_2 = 0$, i.e. at the point of intersection of the plane $S_k$ with the ray. The quantities $p_j$ are the generalized impulses for the ray co-ordinates $q_j$. They represent the components of the slowness vector of point $M$ in the plane $S_k$ with respect to the co-ordinates $q_j$. The function $J$ in $r(s)$ can then be obtained from

\begin{equation}
(5) \quad J = Q_{11}Q_{22} - Q_{12}Q_{21}.
\end{equation}

It has been shown in [2] that by choosing certain new parameters the additional system (4) can be transformed into a system of 5 linear first order ordinary differential equations where one of the functions is $J$. It is our main concern in this paper to show that Eqs (4) can also be easily transformed in such a way that $J$ can be obtained by only integrating a system of three non-linear first order differential equations. The unknown functions of the system have a simple geometrical interpretation. They are the quantities which provide a second order approximation of the wave front.

### 2. THE WAVE-FRONT CURVATURE METHOD

The method described below is based on obtaining $J$ from the following formula [1]:

\begin{equation}
(6) \quad J(s)/J(s_0) = \frac{d\Sigma(s)}{d\Sigma(s_0)} = \exp \int_{s_0}^{s} (1/R_1(s) + 1/R_2(s)) \, ds.
\end{equation}

The integral in (6) is evaluated along the ray between $r(s_0)$ and $r(s)$. $d\Sigma(s)$ and $d\Sigma(s_0)$ are the cross-sectional areas of the ray tube in $r(s)$ and $r(s_0)$, respectively. $R_1(s)$ and $R_2(s)$ are the principal wave-front radii in $r(s)$. Note that formula (6) is only valid as long as the ray connecting $r(s_0)$ with $r(s)$ does not encounter a first order interface along its path. In the latter case one should write

\begin{equation}
(7) \quad J(s)/J(s_0) = \left[ d\Sigma(s)/d\Sigma(S) \right] \left[ d\Sigma(S)/d\Sigma(S) \right] \left[ d\Sigma(S)/d\Sigma(s_0) \right].
\end{equation}

d$\Sigma(S)$ and $d\Sigma(S)$ are the cross-sectional areas of the ray tube on the incident and refracted (reflected) side of the interface, respectively. To the first and third term on the r.h.s. of (7) one can apply formula (6). The middle term in (7), however, is known to be no more than

\begin{equation}
(8) \quad d\Sigma(S)/d\Sigma(S) = \cos \beta/\cos \alpha,
\end{equation}

where $\alpha$ is the incidence and $\beta$ the refraction (reflection) angle. In addition to the co-ordinates $(q_1, q_2, s)$, introduced in [4], we also introduce here a co-ordinate $q_3$. It is assumed to point along the direction of $t$ and has its origin in $q_1 = q_2 = 0$. 