Second-Order Shell Kinematics Implied by Rotation Constraint-Equation

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Abstract. The paper presents a general methodology of introducing the shell-type variables which is based on the rotation constraint-equation (RC-equation). The RC-equation is proven to be equivalent to the polar decomposition of the deformation gradient formula, and the rotations which it yields are interpreted in terms of rotations of vectors of an ortho-normal basis. The deformation function and rotations are assumed as polynomials of the thickness coordinate $\zeta$, and in this form used in the RC-equation. Solving this equation, we can express the coefficients of the quadratic deformation function in terms of the following shell-type variables: (a) the mid-surface position $x_0$, (b) the constant rotation $Q_0$, (c) the rotation vector $\psi^*$ for the $\zeta$-dependent rotations, and (d) the normal components $U_{33}^0$ and $U_{33}^1$ of the right stretching tensor. This new methodology (i) ensures that all shell kinematical variables are consistent with the RC-equation, which is justified on 3D grounds, (ii) provides a general framework from which various Reissner-type hypotheses can be obtained by suitable assumptions. As an example, two generalized Reissner hypotheses are derived: one with two normal stretches, and the other with the in-plane twist and the bubble-like warping parameters.

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1. Introduction

In a classical approach to shells, the dependence of the current position vector $\mathbf{x}$ on the thickness coordinate $\zeta$ is postulated as the so called kinematical hypothesis, such as, e.g., the standard Reissner’s hypothesis,

$$\mathbf{x}(\zeta) = \mathbf{x}_0 + \zeta \mathbf{a}_3, \quad \zeta \in \left[ -\frac{h}{2}, +\frac{h}{2} \right],$$

where $\mathbf{x}_0$ is the position of the middle surface in the deformed configuration, $\mathbf{a}_3 = Q_0 t_3$ is the forward-rotated director $t_3$, and $Q_0 \in SO(3)$ is the rotation tensor. The standard Reissner’s hypothesis can be enhanced in many ways, but when new

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parameters are simply added, then it is difficult to verify their correctness and find their relation to 3D equations.

In the present paper we propose a general methodology of introducing the shell-type mid-surface variables into a 3D formulation, which does not rely on a kinematical hypothesis, but is based on the rotation constraint-equation (RC-equation).

We start from assuming the current position vector \( \mathbf{x} \) and the rotation \( \mathbf{Q} \in SO(3) \) in a general form,

\[
\mathbf{x}(\zeta) = \mathbf{x}_0 + \zeta \mathbf{x}_1 + \zeta^2 \mathbf{x}_2, \quad \mathbf{Q}(\zeta) \approx (\mathbf{I} + \zeta \mathbf{\psi}^* \times \mathbf{I})\mathbf{Q}_0,
\]

where \( \mathbf{x}(\zeta) \) is a second-order polynomial of \( \zeta \), and the \( \zeta \)-dependent form of \( \mathbf{Q} \) is explained in Section 2.1. Then, we use the RC-equation

\[
\text{skew}(\mathbf{Q}^T \mathbf{F}) = 0,
\]

the equivalence of which to the polar decomposition of the deformation gradient \( \mathbf{F} \) is proven in Appendix A, and the interpretation of \( \mathbf{Q} \) given in Appendix B. Besides, we use the right stretch tensor, \( \mathbf{U} \equiv (\mathbf{F}^T \mathbf{F})^{1/2} \), but in the form implied by the RC-equation,

\[
\mathbf{U} = \text{sym}(\mathbf{Q}^T \mathbf{F}),
\]

see also (A.7). With the help of the above two relations we are able to express \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) of (2) in terms of shell-type mid-surface variables, see Section 2.3. The only assumption which we make is that the shell is locally shallow (or very thin), which is not too restrictive, if we consider, e.g., the methodology of deriving four-node shell elements.

As a result, the current position vector \( \mathbf{x}(\zeta) \) is parameterized in terms of the following shell variables: the mid-surface position \( \mathbf{x}_0 \), parameters of the constant rotation \( \mathbf{Q}_0 \), the canonical rotation vector \( \mathbf{\psi}^* \), and the normal stretches \( U_{33}^0 \) and \( U_{33}^1 \). The obtained parametrization is comprehensive, and contains several known kinematical hypothesis, classical and enhanced; some of them are presented and discussed in Section 3.

We note that the proposed methodology allows to avoid introducing the variables which are not consistent with the RC-equation, and that it provides a solid ground for simplifying assumptions made in terms of shell variables.

2. Shell-Type Parametrization of Deformation Function

2.1. GENERAL ASSUMPTIONS

Consider a 3D shell-like body, parameterized by convective coordinates \( \{S^\alpha, \zeta\} \), where \( S^\alpha \) are orthogonal arc-length coordinates on the reference surface, and \( \zeta \in [-h/2, +h/2] \) is the thickness coordinate, see Figure 1. To define the deformation function, \( \chi: \mathbf{x} = \chi(\mathbf{y}) \), which maps the reference configuration onto