AN EQUIVALENT FLAT CONDITION
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Abstract: We show that on an 8-dimensional manifold with Euler characteristic zero every semiflat metric must be flat.

Keywords: 8-dimensional manifold, Euler characteristic zero, semiflat metric, flat

1. Introduction

The local geometry of a manifold provides us with information about its global topology. For instance, the generalized Gauss–Bonnet theorem [1, 2] states that the Euler characteristic $\chi$ of a compact oriented Riemannian manifold $M^{4k}$ can be written as the integral

$$\chi = \frac{2}{V} \left[ \frac{(2k)!}{4k} \right] \int_M \text{trace}(\ast R_{2k} \ast R_{2k}) \, dV$$

where $V$ is the volume of the Euclidean unit $4k$-sphere, $dV$ is the volume element of $M$, $\ast$ is the Hodge $\ast$-operator, and $R_{2k}$ is the $2k$-curvature operator. If $R_{2k}$ commutes with $\ast$, i.e., $R_{2k} \ast = \ast R_{2k}$, we say that the Thorpe condition holds and call the metric a Thorpe metric and the manifold a Thorpe manifold. In the 4-dimensional case every Thorpe metric is Einstein [3, 4]. For dimensions $4k$ higher than 4, Thorpe manifolds were studied in [4]. On the other hand, if $R_{2k}$ anticommutes with $\ast$, i.e., $R_{2k} \ast = - \ast R_{2k}$, then we say that the anti-Thorpe condition holds and call the metric an anti-Thorpe metric and the manifold an anti-Thorpe manifold. From now on, we call a Riemannian metric half-flat if it is both scalar-flat and conformally flat; in other words, if its scalar curvature and Weyl tensor both vanish. In particular, in dimension 4, an anti-Thorpe metric is half-flat, and vice versa. However, in dimension $4k$ higher than 4, an anti-Thorpe metric is not necessarily half-flat, and vice versa. For instance, let $T^{2k+1}$ be a flat torus and let $M^{2k-1}$ be some compact oriented nonflat Riemannian manifold. Then the Riemannian product $T^{2k+1} \times M^{2k-1}$ is an anti-Thorpe manifold. However, in general, this product metric is not half-flat. On the other hand, let $S^{4k}$ be a standard $4k$-sphere and let $H^{4k}$ be a standard $4k$-hyperbolic manifold; then the product metric of $S^{4k}$ and $H^{4k}$ is half-flat. However, this product metric is not anti-Thorpe. We call a Riemannian metric on a compact oriented Riemannian manifold $M^{4k}$ semiflat if it satisfies both the half-flat condition and the anti-Thorpe condition. A semiflat metric is not necessarily flat. For instance, the product metric of $S^{4k+2}$ and $H^{4k+2}$ is semiflat but not flat. The purpose of this article is to see when a semiflat metric is flat.

Theorem 1. On a compact oriented 8-dimensional manifold with $\chi = 0$ every semiflat metric is flat.

The following is a crucial ingredient in the proof of this theorem:

Lemma 1. If $(M, g)$ is a Riemannian manifold of dimension 8 then

$$\text{trace } R_4 = \frac{1}{22} \left( \frac{1}{6} \right) \left( \frac{30}{56} s^2 - \frac{10}{3} |\text{ric}_o|^2 + 4 |W|^2 \right)$$

where $s$ is the scalar curvature, $\text{ric}_o$ is the traceless Ricci tensor, i.e., $\text{ric}_o = \text{ric} - \frac{s}{8} g$ and $W$ is the Weyl tensor.

From Lemma 1 we can observe that trace $R_4$ is nonpositive provided that the metric is half-flat.

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2. The $p$-Curvature Operator

Let $M$ be a Riemannian manifold of dimension $n$, and let $\Lambda^p(M)$ denote the bundle of $p$-vectors of $M$. $\Lambda^p(M)$ is a Riemannian vector bundle, with an inner product on the fiber $\Lambda^p(x)$ above a point $x$ [2]. Let $R$ denote the covariant curvature tensor of $M$. For each even $p > 0$, define the $p$-curvature tensor $R_p$ of $M$ to be the covariant tensor field of order $2p$ given by

$$R_p(u_1, \ldots, u_p, v_1, \ldots, v_p) = \frac{1}{2^{p/2} p!} \sum_{\alpha, \beta \in S_p} \varepsilon(\alpha)\varepsilon(\beta) R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)}) \cdots R(u_{\alpha(p-1)}, u_{\alpha(p)}, v_{\beta(p-1)}, v_{\beta(p)})$$

where $u_i, v_j \in T_x M$, $S_p$ denotes the group of permutations of $(1, \ldots, p)$, and, for $\alpha \in S_p$, $\varepsilon(\alpha)$ is the sign of the permutation $\alpha$.

The tensor $R_p$ has the following properties: it is alternating in the first $p$ variables, alternating in the last $p$ variables, and invariant under the replacement of the first $p$ variables with the last $p$ variables. Hence, at each point $x \in M$, $R_p$ can be regarded as a symmetric bilinear form on $\Lambda^p(x)$. On using the inner product on $\Lambda^p(x)$, $R_p$ at $x$ may then be identified with a selfadjoint linear operator $R_p$ on $\Lambda^p(x)$. Explicitly, this identification is given by

$$(R_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p) \equiv R_p(u_1, \ldots, u_p, v_1, \ldots, v_p)$$

with $u_i, v_j \in T_x M$. From now on, we will use the same notations for the $p$-curvature operators and the $p$-curvature tensors. If $p = n$ then the space $\Lambda^n(x)$ is one-dimensional and hence the selfadjoint linear operator $R_n : \Lambda^n(x) \rightarrow \Lambda^n(x)$ is a scalar multiple of the identity. More explicitly, when expressed globally, the line bundle homomorphism $R_n : \Lambda^n(M) \rightarrow \Lambda^n(M)$ is $R_n = KI$ where $I$ is the identity automorphism of $\Lambda^n(M)$ and $K$ is the Lipschitz–Killing curvature of $M$ [5]. Furthermore, for $x \in M$

$$K(x) = R_n(e_1, \ldots, e_n, e_1, \ldots, e_n)$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis for $T_x M$.

The generalized Gauss–Bonnet theorem [1] expresses the Euler characteristic $\chi$ of a compact oriented Riemannian manifold of even dimension $n$ as the integral

$$\chi = \frac{2}{c_n} \int_M K dV$$

where $K$ is the Lipschitz–Killing curvature of $M$, $c_n$ is the volume of the unit Euclidean $n$-sphere, and $dV$ is the volume element of $M$.

We see now that the Lipschitz–Killing curvature $K$ of $M$ can be expressed in terms of $R_p$ and the Hodge $*$-operator.

If $M$ be an oriented Riemannian manifold of even dimension $n$, then according to [2] the Lipschitz–Killing curvature $K$ of $M$ can also be expressed as the function whose value at $x \in M$ is

$$\frac{p!(n-p)!}{n!} \operatorname{trace}(R_{n-p} \ast R_p),$$

where $p = 2, 4, 6, \ldots, (n-2)$. For an oriented Riemannian manifold of dimension $n = 4k$, we can consider the middle curvature operator $R_{2k}$, and if this operator satisfies the anti-Thorpe condition,

$$R_{2k} \ast = - \ast R_{2k},$$

then, since $\ast^2 = \operatorname{Id}$, the trace formula for $K$ reduces to

$$K = -\frac{[(2k)]^2}{(4k)!} \operatorname{trace} R_{2k}^2 \leq 0$$

where equality holds if and only if $R_{2k} = 0$. From the above facts, we can easily infer a necessary condition for the existence of an anti-Thorpe metric.