On the Stability of Bifurcation Diagrams of Vanishing Flattening Points

R. Uribe-Vargas

Received May 13, 2002

ABSTRACT. On a smooth surface in Euclidean 3-space, we consider vanishing curves whose projections on a given plane are small circles centered at the origin. The bifurcations diagram of a parameter-dependent surface is the set of parameters and radii of the circles corresponding to curves with degenerate flattening points.

Solving a problem due to Arnold, we find a normal form of the first nontrivial example of a flattening bifurcation diagram, which contains one continuous invariant.

KEY WORDS: flattening point, bifurcation diagram, singularity of a family of mappings.

1. Introduction. Recently, Arnold studied small perturbations of strongly degenerate Lagrangian and Legendre mappings (vanishing singularities) and obtained various modifications of the classical four-vertex theorem for a closed convex plane curve (see [2–4] and also [6,7]). In particular, new lower bounds for the number of flattening points of space curves [2] were found.

Here we solve one of Arnold’s problems in the field [5].

A point on a space curve where the torsion vanishes (the first three derivatives of the curve are linearly dependent) is called a flattening point. A degenerate (multiple) flattening point is a multiple zero of the torsion.

We are interested in flattening points of a family of (vanishing) space curves \( \gamma(f,c): S^1 \rightarrow \mathbb{R}^3 = \{x,y,w\} \) defined as the intersection of a smooth surface \( w = f(x,y) \) in \( \mathbb{R}^3 \) (which is the graph of a function \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \)) with the family \( x^2 + y^2 = c \) of coaxial cylinders whose radius \( r = \sqrt{c} \) tends to zero. Subtracting an affine function from \( f \) does not affect the projections of the flattening points, and so we can always assume that the differential of \( f \) at the origin is zero.

If the second differential \( d^2 f_0 \) at zero is nonumbilic (i.e., is not a multiple of \( dx^2 + dy^2 \)), then the curve \( \gamma(f,c) \) has exactly four nondegenerate flattening points for sufficiently small \( c \).

In the three-dimensional space of quadratic forms in two variables, the set of umbilic forms has the codimension 2, and so an umbilic degeneracy of \( d^2 f_0 \) can occur in generic families of functions only if there are at least two parameters.

The bifurcation diagram \( B(F) \) of a family of functions \( F: \mathbb{R}^2 \times \Lambda \rightarrow \mathbb{R} \) depending on parameters \( \lambda \in \Lambda = \mathbb{R}^k \) is the set of pairs \( (\lambda,c) \in \Lambda \times \mathbb{R} \) such that the curve \( \gamma(F(\cdot,\lambda),c) \) has a degenerate flattening point.

We study the singularities of the bifurcation diagram at \( c = 0 \) of a generic two-parameter family \( F \) such that \( f = F|_{\lambda=0} \) is a function with an umbilic critical point at the origin up to a so-called polar equivalence of families of functions, which preserves the diffeomorphic type of bifurcation diagrams.

We show that the 3-jet of such a family (with parameters \( a, b \)) is equivalent to the jet of the polynomial family \( f_0 = x^3 - 3xy^2 + 2bxy + a(x^2 - y^2) \), whose bifurcation diagram is a parabolic surface (cup) with six cuspidal edges meeting at the origin (see Fig. 1).

The section of this surface by a coordinate plane \( c = \text{const} \) is a six-cusped hypocycloid (the trajectory of a point of a circle of radius \( r = \sqrt{c} \) rolling without slipping on the inner side of a fixed circle of radius \( 6r \)).

The bifurcation diagram of any extension of \( f_0 \) by arbitrary higher-order terms tends to that of \( f_0 \) as \( r \rightarrow 0 \) (Proposition 2). The parameters of points in the interior of the cup correspond
to curves with six simple flattening points, and the points in the exterior correspond to curves with four flattening points. Regular points of the surface itself correspond to curves with double flattening points, and those on the cuspidal edges define curves with triple flattening points.

Our main result (Theorem 1) is the proof of the fact that, up to polar equivalence, generic $C^\infty$ umbilic families of functions have exactly one modulus (continuous invariant of the orbit). We use a technique close to [1].

**Acknowledgements.** The author expresses his gratitude to V. I. Arnold for setting the problem as well as to V. Zakalyukin and M. Kazarian for help.

2. Definitions and Results. By $L$ we denote the derivation along the Hamiltonian vector field $V_h = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ on $\mathbb{R}^2$, which is tangent to the circles $S_c: x^2 + y^2 = c$.

**Proposition 1.** The flattening points of the curve $\gamma(f, c)$, $c > 0$, are the zeros of the restriction of the function $L(L^2 + 1)f$ to $S_c$.

The proof is by a straightforward computation of the volume form on the first three derivatives of the curve parametrized by the polar angle $\varphi$. In the polar coordinates, one has $L = \frac{\partial}{\partial \varphi}$.

Let $\hat{f} = L(L^2 + 1)f$ be the image of $f$ under the operator $L(L^2 + 1)$. The degenerate flattening points are the common zeros of $\hat{f}$ and $L\hat{f}$. Hence families of functions having the same image under the operator $L(L^2 + 1)$ have the same bifurcation diagram.

Consider a family $F: \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$ of functions depending on parameters $\lambda \in \mathbb{R}^k$. The family $\hat{F} = L(L^2 + 1)F$ will be called the generating family of the bifurcation diagram $B(F) = \{(\lambda, c) | \exists (x, y) : \hat{F} = L\hat{F} = 0, h = 0\}$, where $h(x, y, c) = x^2 + y^2 - c$. The family of mappings $H_{\hat{F}}: \mathbb{R}^2 \times \mathbb{R}^k \times \mathbb{R}_+ \to \mathbb{R}^2$, $H_{\hat{F}}: (x, y, \lambda, c) \mapsto (\hat{F}, h)$, depending on an additional parameter $c \in \mathbb{R}$, $c \geq 0$, will be called the defining mapping.

**Definitions.** Two families $F_1, F_2: \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$ are said to be cylindrically equivalent if their defining mappings are contact equivalent, in other words, if there exists a nonsingular $2 \times 2$ matrix $M$ whose entries are smooth functions in $(x, y, \lambda, c)$ and a diffeomorphism $\Theta: \mathbb{R}^2 \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}^k \times \mathbb{R}$ that preserves the boundary $c = 0$ and the fibration $\pi: (x, y, \lambda, c) \mapsto (\lambda, c)$ over the parameter space (i.e., satisfies $\pi \circ \Theta = \Theta' \circ \pi$ for some diffeomorphism $\Theta': \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k \times \mathbb{R}$ of the parameter space) such that

$$H_{F_2} \circ \Theta = M \cdot H_{F_1}.$$  

Two families $F_1, F_2: \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}$ are said to be polar-equivalent if their images $\hat{F}_1, \hat{F}_2$ are cylindrically equivalent.

**Remarks.** 1. Clearly, cylindrically equivalent generating families (and hence polar-equivalent families) determine diffeomorphic bifurcation diagrams.

2. The cylindrical equivalence orbits of the families $F + \phi(x^2 + y^2 - c)$ in $x$, $y$, $\lambda$, $c$ with arbitrary functions $\phi$ coincide. Thus in the sequel we identify families in $x$, $y$ with parameters $a$, $b$, $c$ differing by a function that belongs to the ideal generated by the equation $x^2 + y^2 - c = 0$.

In the complex variables $z = x + iy$, $\bar{z} = x - iy$ on $\mathbb{R}^2$, the operator $L$ acquires the form $L = i (z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}})$. Each monomial $z^m \bar{z}^n$ is an eigenfunction of $L(L^2 + 1)$ with eigenvalue $((n -