Computing the Distribution of the Maximum of Gaussian Random Processes

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Abstract. The aim of this paper is to propose an Splus program to calculate bounds for the distribution of the maximum of a smooth Gaussian process on a fixed interval. We generalize the results given in Azais et al. (1999) to the case of the absolute value of the Gaussian process and to the non-homogeneous case. Our method relies on calculations of the first three terms of the Rice’s series. Some applications are given to illustrate the method and the performances of the program. The corresponding Splus functions are available at the URL: http://www.lsp.ups-tlse.fr/cedelmas/software.html.

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1. Introduction

Let \( X = \{ X(t); t \in [0,T] \} \) be a Gaussian random process. The evaluation of the tail probability:

\[
P \left( \max_{t \in [0,T]} X(t) \geq u \right) \quad \text{or} \quad P \left( \max_{t \in [0,T]} |X(t)| \geq u \right)
\]

for a fixed \( u \in \mathbb{R} \), is a key point in many statistical problems. For some examples, in parametric statistics, see Davies (1977, 1987) and Dacunha-Castelle and Gassiat (1997, 1999); in nonparametric statistics, see Sun (1991), Chaudhuri and Sengupta (1993), Piterbarg and Tyurin (1993), Park and Sun (1998), and Chaudhuri and Marron (1999, 2000); in image processing, see Worsley et al. (1992) and Adler (2000); in genetical problems, see Cierco (1998) and Azais and Cierco-Ayrrolles (2002). In most cases the tail probability (1) occurs in testing situations. Classical methods can be used to evaluate (1) such as:

(M1) The Davies’ bound (Davies, 1977, 1987) that is an upper bound for (1) and that represents a one-term approximation for this tail probability.
(M2) A Monte-Carlo method where the sample paths of $X$ are simulated by classical methods described for example in Azaïs (1990).

Our approach will be the Rice’s method which is, generally speaking, more accurate than other methods to evaluate (1). It has been introduced and studied especially by Kac (1943), Rice (1944–1945), Cramér and Leadbetter (1967), Wschebor (1985), Azaïs and Wschebor (1997) and Azaïs et al. (1999). The aim of this paper is:

- to generalize the method and results given in Azaïs et al. (1999) for an homogeneous Gaussian process to the non-homogeneous case and to the case of the absolute value of the Gaussian process,
- to propose Splus functions to obtain the required approximations for the tail probability (1), for any wished $u > 0$, and for the threshold at any wished level.

Further analysis of the method is given in Section 2. In Section 3 the bounds are derived. To illustrate the performances of the Splus program we give some major examples in Section 4. The Splus functions of the program are available at the URL: http://www.lsp.ups-tlse.fr/Cdelmas/software.html.

2. The Rice’s Method

Let $X$ be a Gaussian process with almost surely $\mathcal{G}^1$ sample paths on a fixed interval $[0, T]$. Let $u$ be a real number. The number of upcrossings (respectively downcrossings) of the level $u$ by $X$, denoted by $U^X_u[0, T]$ (respectively $D^X_u[0, T]$), is defined as follows:

$$U^X_u[0, T] = \# \{ t \in [0, T] : X(t) = u, X'(t) > 0 \}$$

$$D^X_u[0, T] = \# \{ t \in [0, T] : X(t) = u, X'(t) < 0 \}.$$

We have:

$$P \left[ \sup_{t \in [0, T]} X(t) > u \right] = P[X(0) > u] + P \left[ U^X_u[0, T]_{|X(0)| \leq u} \geq 1 \right],$$

$$P \left[ \sup_{t \in [0, T]} |X(t)| > u \right] = P \left[ |X(0)| > u \right] + P[(U^X_u[0, T] + D^X_u[0, T])_{|X(0)| \leq u} \geq 1].$$

Then, by analogous arguments as those involved in Azaïs et al. (1999), we easily obtain:

$$P[X(0) > u] + E[U^X_u[0, T]_{|X(0)| \leq u}] - \frac{1}{2} E[U^X_u[0, T]^{[2]}] \leq P[\sup_{t \in [0, T]} X(t) > u]$$

$$\leq P[X(0) > u] + E[U^X_u[0, T]_{|X(0)| \leq u}]$$

and:

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