International Journal of Theoretical Physics, Vol. 38, No. 3, 1999

A Topos Perspective on the Kochen–Specker Theorem II. Conceptual Aspects and Classical Analogues

J. Butterfield[1] and C. J. Isham[2]

Received September 2, 1998

In a previous paper, we proposed assigning as the value of a physical quantity in quantum theory a certain kind of set (a sieve) of quantities that are functions of the given quantity. The motivation was in part physical—such a valuation illuminates the Kochen–Specker theorem—and in part mathematical—the valuation arises naturally in the topos theory of presheaves. This paper discusses the conceptual aspects of this proposal. We also undertake two other tasks. First, we explain how the proposed valuations could arise much more generally than just in quantum physics; in particular, they arise as naturally in classical physics. Second, we give another motivation for such valuations (that applies equally to classical and quantum physics). This arises from applying to propositions about the values of physical quantities some general axioms governing partial truth for any kind of proposition.

1. INTRODUCTION

In a previous paper [1]—referred to hereafter as (I)—we proposed assigning as the value of a physical quantity in quantum theory a certain kind of set (a sieve) of quantities that are functions of the given quantity. Our motivation was in part physical—such a valuation illuminates the Kochen–Specker theorem—and in part mathematical—the valuation arises naturally in the topos theory of presheaves. These aspects were closely linked. We interpreted a valuation as assigning truth values to propositions '$A \in \Delta$' asserting that the value of the quantity A lies in the Borel subset Δ of the spectrum of the operator $\hat{A}$ that represents A. The fact that one quantity can

[1] All Souls College, Oxford OX1 4AL, U.K.; e-mail: jb56@cus.cam.ac.uk; jeremy.butterfield@all-souls.oxford.ac.uk.
[2] The Blackett Laboratory, Imperial College of Science, Technology & Medicine, London SW7 2BZ, U.K.; e-mail: isham@ic.ac.uk.

0020-7748/99/0300-082 7$16.00/0 © 1999 Plenum Publishing Corporation

be a function, or coarse-graining, of another implies that there is a natural presheaf associated with these propositions. And the theory of presheaves gives a natural generalization of the *FUNC* property (viz., that the value of a function of a given quantity is the function of the value of the quantity), which plays a central role in the Kochen–Specker theorem.

In this paper, we first show how sieve-valued valuations obeying our generalization of *FUNC* arise much more generally than just as the values of quantities in quantum physics, and, accordingly, how the principal results of (I) can be generalized. In fact, we claim that they are one of the most natural notions of valuation for any presheaf of propositions, no matter what their topic. From a physical perspective, a mathematical structure of this type is indicated whenever the idea of 'contextual' statements about the system (i) is physically appropriate and (ii) is so in such a way that the set of all possible such contexts can be regarded as the objects in a category, which then forms the base category over which the presheaves are defined. As we shall see, in making this claim we assume about valuations on propositions only the basic idea that they must be some sort of structure-preserving function from the set of propositions (with operations such as negation, conjunction, etc., defined on it) to the set of truth values, which is to be some sort of logical algebra.

That is the task of Section 3, where we show that sieve-valued valuations arise naturally in classical physics, and Section 4, where we show how such valuations can arise in even more general contexts. But first, to facilitate reading the paper, there is a short review of the elements of the theory of presheaves [more concise than in (I), but with some extra heuristic material], and of how these ideas were applied in (I) to quantum physics.

The paper ends with a presentation of another motivation for such valuations (Section 5). We will argue that intuitive ideas about what might be meant by the notion of 'partial truth' (applying to any type of proposition) make sieve-valued valuations very natural. Among these principles, the main one will be that a proposition is nearer to 'total truth,' the larger the subset of its consequences that are themselves totally true. This argument, and the principles it refers to, is conceptual, not mathematical: indeed, it will not need the mathematical notions of Section 2, except the idea of a category—that *is* compulsory, in order to make sense of the notion of a sieve. But the argument and its principles can be made precise most naturally by using the ideas of presheaf theory; in particular, the idea of 'consequence' (entailment) can be made precise in terms of the generalized notion of coarse-graining introduced in Section 4. Again, we shall see that the proposals of (I) arise from applying these general ideas to propositions about the values of quantum physical quantities.