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Topos Perspective on the Kochen–Specker Theorem: I. Quantum States as Generalized Valuations

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Any attempt to construct a realist interpretation of quantum theory founders on the Kochen–Specker theorem, which asserts the impossibility of assigning values to quantum quantities in a way that preserves functional relations between them. We construct a new type of valuation which is defined on all operators, and which respects an appropriate version of the functional composition principle. The truth-values assigned to propositions are (i) contextual and (ii) multivalued, where the space of contexts and the multivalued logic for each context come naturally from the topos theory of presheaves. The first step in our theory is to demonstrate that the Kochen–Specker theorem is equivalent to the statement that a certain presheaf defined on the category of self-adjoint operators has no global elements. We then show how the use of ideas drawn from the theory of presheaves leads to the definition of a generalized valuation in quantum theory whose values are sieves of operators. In particular, we show how each quantum state leads to such a generalized valuation. A key ingredient throughout is the idea that, in a situation where no normal truth-value can be given to a proposition asserting that the value of a physical quantity A lies in a subset $\Delta \subseteq \mathbb{R}$, it is nevertheless possible to ascribe a partial truth-value which is determined by the set of all coarse-grained propositions that assert that some function $f(A)$ lies in $f(\Delta)$, and that *are* true in a normal sense. The set of all such coarse-grainings forms a sieve on the category of self-adjoint operators, and is hence fundamentally related to the theory of presheaves.

1. INTRODUCTION

1.1. Preliminary Remarks

Anyone who has taught an introductory course on quantum theory will have encountered the anguished disbelief that can accompany a student's

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first encounter with the problematic status of beliefs previously deemed to be self-evidently true. In particular, it is difficult to remove the compelling belief that, at any given time, any physical quantity must have a value.

In classical physics, there is no problem with this belief since the underlying mathematical structure is geared precisely to express it. Specifically, if $\mathcal{S}$ is the state space of some classical system, a physical quantity A is represented by a real-valued function $\overline{A}$: $\mathcal{S} \to \mathbb{R}$; and then the value $V_s(A)$ of A in any state $s \in \mathcal{S}$ is simply

$$V^s(A) = \overline{A}(s) \tag{1.1}$$

Thus all physical quantities possess a value in any state. Furthermore, if h: $\mathbb{R} \to \mathbb{R}$ is a real-valued function, a new physical quantity $h(A)$ can be defined by requiring the associated function $\overline{h(A)}$ to be

$$\overline{h(A)}(s) := h(\overline{A}(s)) \tag{1.2}$$

for all $s \in \mathcal{S}$; i.e., $\overline{h(A)} := h \circ \overline{A}$: $\mathcal{S} \to \mathbb{R}$. Thus the physical quantity $h(A)$ is *defined* by saying that its value in any state s is the result of applying the function h to the value of A; hence, by definition, the values of the physical quantities $h(A)$ and A satisfy the 'functional composition principle'

$$V^s(h(A)) = h(V^s(A)) \tag{1.3}$$

for all states $s \in \mathcal{S}$.

However, to the distress of angst-ridden students, standard quantum theory precludes any such naive realist interpretation of the relation between formalism and physical world. And this is not just because of some wilfully obdurate philosophical interpretation of the theory: rather, the obstruction comes from the mathematical formalism itself, in the guise of the famous Kochen–Specker theorem, which asserts the impossibility of assigning values to all physical quantities while at the same time preserving the functional relations between them (Kochen and Specker, 1967).[3]

In a quantum theory, a physical quantity A is represented by a self-adjoint operator $\hat{A}$ on the Hilbert space of the system, and the first thing one has to decide is whether to regard a valuation as a function of the physical quantities themselves, or on the operators that represent them. From a mathematical perspective, the latter strategy is preferable, and we shall therefore define a (global) valuation to be a real-valued function V on the set of all bounded, self-adjoint operators, with the properties that (i) the value $V(\hat{A})$ of the physical quantity A represented by the operator $\hat{A}$ belongs to the spectrum

[3] As has been emphasized by Brown (1992), the essential result is already contained in Bell's seminal first paper on hidden variables (Bell, 1987).