LOCALIZED COHERENT STRUCTURES IN THE BOUNDARY LAYER

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A Blasius laminar boundary layer and a steady turbulent boundary layer on a flat plate in an incompressible fluid are considered. The spectral characteristics of the Tollmien–Schlichting (TS) and Squire waves are numerically determined in a wide range of Reynolds numbers. Based on the spectral characteristics, relations determining the three-wave resonance of TS waves are studied. It is shown that the three-wave resonance is responsible for the appearance of a continuous low-frequency spectrum in the laminar region of the boundary layer. The spectral characteristics allow one to obtain quantities that enter the equations of dynamics of localized perturbations. By analogy with the laminar boundary layer, the three-wave resonance of TS waves in a turbulent boundary layer is considered.

Introduction. Based on a large amount of experimental data, physical processes in laminar and turbulent boundary layers were analyzed in [1–4]; special attention was paid to coherent structures, which may be related to the linear and nonlinear dynamics of wave packets. To describe the wave packets, one has to use the Navier–Stokes equations in the laminar boundary layer and the Reynolds equations and equations for oscillations in the turbulent boundary layer. Following [5], these equations can be represented in the form of a system of inhomogeneous Orr–Sommerfeld and Squire equations equivalent to the initial problem. Methods of identification of the special features of these flows were proposed in [6–8] for obtaining simpler equations. A number of assumptions were used: isotropy of the phase velocity, linearity of the phase velocity in terms of the absolute value of the wave vector \( k \), finiteness of the imaginary part of the frequency of TS waves with the streamwise wavenumber tending to zero, and nonmonotonic dependence of the phase velocity on \( k \).

In the case of a laminar boundary layer, the equations reduce to a system of Schrödinger nonlinear equations with respect to the envelopes of the wave packets \( \tilde{\psi}_n(0) \) related by harmonic and three-wave resonances, which are supplemented by a nonlinear integrodifferential equation with respect to the amplitude \( \psi(0) \) of the wave packet concentrated near the origin of the space of wavenumbers [6]. In the case of zero amplitude \( \tilde{\psi}_n(0) \), the equation for \( \psi(0) \) retains its own significance and has the form

\[
\frac{\partial}{\partial t} \psi(0)(t, r_1) - \varepsilon(\bar{X}_0 - a(\bar{X}_0)) \frac{\partial}{\partial x_1} \psi(0)(t, r_1) = \varepsilon^2 I(0),
\]

\[
I(0) = -b(\bar{X}_0) \int \frac{1}{|r_1 - s|} \frac{\partial}{\partial s} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \psi(0)(t, s) \, ds
\]

\[
- i x_1 \frac{\partial a(\bar{X}_0)}{\partial \bar{X}_0} \frac{\partial}{\partial x_1} \psi(0)(t, r_1) + (d(\bar{X}_0) + Q(0)) \psi(0)(t, r_1) + H_{0,0}(\bar{\psi}(0)(t, r_1)),
\]

where \( r_1 = r/\varepsilon, \ r = (x, z), \ r_1 = (x_1, z_1), \ \varepsilon^2 = (\nu \tilde{\omega}_{\text{max}}/u_\infty^2)^{1/2}, \ \nu \) is the kinematic viscosity, \( u_\infty \) is the free-stream velocity, \( \tilde{\omega}_{\text{max}} = \max \text{Imag} [\omega(k)], \) and \( \omega \) is the eigenfrequency of the unstable mode of the Orr–Sommerfeld equation. The equation for \( \psi^{(0)} \) contains the quantities \( a(X_0), b(X_0), \) and \( d(X_0) \) determined by the dispersion characteristic of the unstable mode of TS waves; \( Q(0) \) and \( H_{0,0} \) are expressed in terms of the quadratures of combinations of the eigenfunctions of the solutions of the spectral problems for the Orr–Sommerfeld and Squire equations \( [6] \) (they are considered below in more detail), and \( X_0 \) is the coordinate of the “center of mass” of the wave packet normalized to the length scale \( L = u_\infty/\tilde{\omega}_{\text{max}} \). The finiteness of these parameters would indicate the applicability of this model for the description of a set of phenomena, at least weakly linear ones.

In the case of a turbulent boundary layer, we consider models based on the kinetic equation for elementary waves in the three-wave resonance approximation. The coefficients of this equation are also determined from the solution of the spectral problem.

Because of the complexity of determining these quantities, in particular, matrix elements \( H_{k,k_1} \), simplified equations were derived on the basis of the asymptotic analysis of the spectral problem for the Orr–Sommerfeld equation for \( \alpha \text{Re} \to \infty \) (\( \alpha \) is the streamwise wavenumber and Re is the Reynolds number).

Below we give the results of numerical calculation of the spectral characteristics for the Blasius profile and for a self-similar turbulent profile \( [9, 10] \), which are necessary to justify the approaches of \( [6–8] \).

1. Formulation of the Problem. The authors of \( [6–8] \) considered some possibilities of the nonlinear description of the flow in the laminar and turbulent regions of the boundary layer on a flat plate using flowfield decomposition into a series in eigenfunctions of the Orr–Sommerfeld equation. In the general case, one also has to consider the Squire equation, since the Navier–Stokes equations for the fields of velocity \( u, v, w \) and pressure \( p \) may be represented in the form of a system of Orr–Sommerfeld and Squire equations relative to the vertical components of velocity \( v \) and vorticity \( \eta \) with nonlinear right parts \( [5] \). The linear parts of the equations with the boundary conditions \( \hat{\nu} = d\hat{\nu}/dy = \hat{\eta} = 0, \ y = 0, \) and \( y = \infty \) yield the known spectral problems \( [5] \). The quantities marked by the hat symbol are the Fourier transforms of the initial quantities:

\[
f = \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\omega \hat{f}(y) \exp (i\alpha x + i\beta z - i\omega t), \quad \omega = \alpha c.
\]

Here \( k^2 = \alpha^2 + \beta^2, \) \( \beta \) is the spanwise wavenumber, \( k = (\alpha, \beta) \) is the wave vector, and \( c \) is the phase velocity. By solving the Orr–Sommerfeld equation, we obtain the eigenvalues (we call them modes) \( c_n = c_n(k^2, \alpha \text{Re}), \) where \( n = 1, 2, 3, \ldots \) is the number of the eigenvalue.

The right part of the Squire equation contains a term proportional to \( \hat{\nu} \); in the absence of this term, we obtain an equation corresponding to the spectrum of eigenvalues (modes) \( [5] \):

\[
c'_n = c'_n(k^2, \alpha \text{Re}) = c''_n(\alpha \text{Re}) - i \frac{k^2}{\alpha \text{Re}}, \quad n = 1, 2, 3, \ldots .
\]

We indicate some conditions of symmetry \( [5] \) imposed on the phase velocities \( c(k^2, \alpha \text{Re}) \) and \( c''(\alpha \text{Re}) \) and eigenfunctions \( \hat{\nu}(k, \text{Re}) \) and \( \hat{\eta}(k, \text{Re}) \) by the Squire and Orr–Sommerfeld equations and the conditions of reality of the initial physical quantities:

\[
c(k^2, \alpha \text{Re}) = c(k^2, -\alpha \text{Re})^*, \quad c''(\alpha \text{Re}) = c''(-\alpha \text{Re})^*, \quad \hat{\nu}(k) = \hat{\nu}(-k)^*, \quad \hat{\eta}(k) = \hat{\eta}(k)^*.
\]

The asterisk here denotes complex conjugation. It is convenient to use these conditions for constructing resonance characteristics.

We give the first modes of the Squire and Orr–Sommerfeld equations and also some characteristics determined through them, which refer to resonant interactions of elementary waves, i.e., to the resonance of the Squire and Orr–Sommerfeld modes,

\[
c_n(k^2, \alpha \text{Re}) = c'_m(k^2, \alpha \text{Re}), \quad m, n = 1, 2, 3, \ldots ,
\]

(1.1)