Differentiation for Orders and Artinian Rings

Dedicated to Daniel Simson on the occasion of his 60th birthday

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Abstract. The method of differentiation for the category $\Lambda$-lat of lattices over an order $\Lambda$ will be extended to integral almost Abelian categories $\mathcal{A}$ instead of $\Lambda$-lat. In particular, this yields a differentiation for finitely generated left modules over left Artinian rings.


Key words: differentiation, localization, order, Artinian ring.

Introduction

Some powerful methods of representation theory are based on equivalences between categories related to module categories. Whereas tilting modules and so-called $\star$-modules provide equivalences between full subcategories of module categories (e.g., [6–9, 12, 24]), the Green correspondence [17] and its generalization [4] give equivalences between certain quotient categories. In all these cases, the equivalence is induced by a pair of adjoint functors.

In [21] we established an equivalence of the form

$$\tilde{\partial}_u: \Lambda$-lat$/[H] \longrightarrow \Lambda'$-lat$/$[B],$$

where $\Lambda, \Lambda'$ are orders over a complete discrete valuation domain $R$, and $\Lambda$-lat denotes the category of $\Lambda$-lattices. Here the correspondence is functorial only in one direction. It is defined in terms of a hereditary morphism $u: P \hookrightarrow I$ in $\Lambda$-lat. In the most important case where the $\Lambda$-lattices between $P$ and $I$ form a chain $P = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = I$, this means that $P$ is projective and $I$ injective in $\Lambda$-lat, and $H := H_0 \oplus \cdots \oplus H_n$ satisfies

$$\text{Hom}_\Lambda(P, I/P) = \text{Ext}_\Lambda(I/P, I) = \text{Ext}_\Lambda(H, H) = 0.$$ 

Then $B$ is indecomposable projective and injective in $\Lambda'$-lat. Special cases of (0) are Zavadskij’s differentiation algorithms for representations of posets [29] and
tiled orders [30], and Simson’s differentiation algorithm [19, 25, 26] for socle-projective modules over peak algebras. If $K$ is the quotient field of $R$, then $K \otimes_R \Lambda' \cong M_2(K \otimes_R \Lambda)$, and the equivalence (0) is induced by a functor $\delta_u: \Lambda\text{-lat} \to \Lambda'\text{-lat}$, called differentiation. In [22] we impose a weaker condition on $u$ with the effect that $\Lambda'\text{-lat}$ in (0) has to be replaced by a full subcategory. Then $u$ is called prehereditary. In this case, (0) implies a general version of Simson’s splitting theorem ([22], Theorem 4; cf. [27], Theorem 17.53).

In the present article, we extend the equivalence (0) to a very general class of categories instead of $\Lambda\text{-lat}$. In [23] we call these categories integral almost Abelian. They occur in various parts of representation theory, functional analysis, and topological algebra (see §1, and [23], §2). For example, if $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory in an Abelian category $\mathcal{C}$, then the full subcategory $\mathcal{F}$ of $\mathcal{C}$ is integral almost Abelian. Conversely, every integral almost Abelian category $\mathcal{A}$ arises in this way. This description, however, does not reveal the self-dual nature of such categories $\mathcal{A}$.

Let $\mathcal{A}$ be integral almost Abelian. A morphism $p: P^a \to A$ is said to be a $P$-cover if every morphism $P \to A$ factors through $p$. An $I$-hull is defined in a dual way. We call a morphism $u: P \to I$ closed if $u$ is regular (i.e. monic and epic), every object $A$ has a $P$-cover and an $I$-hull, and $u$ itself is a $P$-cover and an $I$-hull. Let $\text{reg}(\mathcal{A})$ denote the category of regular morphism $r: A_1 \to A_0$ in $\mathcal{A}$ modulo homotopy. Then $\text{reg}(\mathcal{A})$ is integral almost Abelian (Proposition 4), and $u$ can be regarded as an object $\overline{u} \in \text{reg}(\mathcal{A})$. By $ab[\overline{u}]$ we denote the full subcategory of objects $\overline{b} \in \text{reg}(\mathcal{A})$ such that some $u^a: P^n \to I^n$ satisfies $u^a = abc$ with regular $a, c \in \mathcal{A}$. A closed morphism $u$ will be called prehereditary if $ab[\overline{u}]$ is Abelian. This concept is related to a weak form of localization (Theorem 1). If $u$ is prehereditary, then $ab[\overline{u}]$ is contained in a full subcategory $\text{ab}[\overline{u}]$ of $\text{reg}(\mathcal{A})$ which is integral and almost Abelian. Furthermore, $\text{ab}[\overline{u}]$ can be recovered from $\text{ab}[\overline{u}]$ by localization, making regular morphisms invertible. Now let $\text{Reg}_{ab}(\mathcal{A})$ be the category of regular morphisms $a: G \to F$ in $\mathcal{A}$ such that $\overline{a} \in \text{ab}[\overline{u}]$, every morphism $P \to G$ extends along $u$ to a morphism $I \to F$, and every $F \to I$ restricts to a morphism $G \to P$. Then we prove a generalization of (0) with $\Lambda\text{-lat}$ replaced by $\mathcal{A}$ and $\Lambda'\text{-lat}$ replaced by $\text{Reg}_{ab}(\mathcal{A})$ (Theorem 2). If $P$ is projective and $I$ injective in $\mathcal{A}$, we call $u: P \to I$ hereditary. In this case, $\text{Reg}_{ab}(\mathcal{A})$ is again integral almost Abelian (Proposition 9).

As a first application, we obtain a global version of (0) for orders $\Lambda$ over a Dedekind domain. Secondly, we get a differentiation for left Artinian rings $\Lambda$ (Theorem 3). For a semisimple left $\Lambda$-module $S$, the category $\Lambda\text{-lat} := \Lambda\text{-lat}\{S\}$ of finitely generated left $\Lambda$-modules $M$ with $\text{Hom}_\Lambda(S, M) = 0$ is integral almost Abelian. If $u: P \Leftarrow I$ is hereditary in $\Lambda\text{-lat}$, there exists a left Artinian ring $\Lambda'$ and a semisimple left $\Lambda'$-module $S'$ such that an equivalence (0) holds for $\Lambda'\text{-lat} := \Lambda'\text{-lat}\{S'\}$. Note that in case $S$ is indecomposable injective, $\Lambda\text{-lat}\{S\}$ consists of the finitely generated $\Lambda$-modules which do not possess a direct summand isomorphic to $S$. 

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