EXTREMAL VALUED FIELDS

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It is shown that every finite-dimensional skew field whose center is an extremal valued field is defect free. We construct an example of an algebraically complete valued field such that a finite-dimensional skew field over it has a non-trivial defect, that is, there exist algebraically complete valued fields that are not extremal.

The reader is assumed to be familiar with fundamentals of the valuation theory of fields within the extent of Chapter 1 in [1]. Below we also use the notation adopted in [1].

Let \( F = \langle F, R \rangle \) be a valued field; we refer to it as extremal if the set \( \{ v_R f(\bar{a}) \mid \bar{a} = a_0, \ldots, a_{n-1} \in F \} \subseteq \Gamma_R \cup \{ \omega \} \) has a greatest element, for any polynomial \( f(x) \in F[x] = F[x_0, \ldots, x_{n-1}] \).

Remark. If, in the definition above, we limit ourselves to polynomials in one variable we will be faced up to a corresponding concept of an algebraically maximal valued field (cf. [1, 2]), that is, a valued field having no proper algebraic immediate extensions. This, in particular, implies that every extremal valued field is Henselian.

**Proposition 1.** Let \( F = \langle F, R \rangle \) be an extremal valued field and \( F \leq F_0 = \langle F_0, R_0 \rangle \) be its finite algebraic extension. Then \( F_0 \) is an extremal valued field.

**Proof.** Let \( [F_0 : F] = m \) and \( \bar{b} = b_0, \ldots, b_{m-1} \) be a basis for \( F_0 \) over \( F \); \( g(y_0, \ldots, y_{m-1}) \) is the norm form of \( F_0 \) over \( F \) w.r.t. \( \bar{b} \) \((g(\bar{y}) \text{ is the norm } N_{F_0/F}(\sum_{i<m} b_i y_i) \text{ of a "common" element } \sum_{i<m} b_i y_i \text{ in } F_0)\). Let \( f(\bar{x}) \in F_0[\bar{x}] = F_0[x_0, \ldots, x_{n-1}] \) and \( F(\bar{z}) \) be a polynomial obtained by substituting in \( f \) the form \( \sum_{i<m} b_i z_i \) for a variable \( x_j \). The polynomial \( F(\bar{z}) \) can be (uniquely) represented as \( F(\bar{z}) = \sum_{i<m} b_i F_i(\bar{z}), \) where \( F_i(\bar{z}) \in F[\bar{z}] \), \( i < m \). Let \( G(\bar{y}) \) be a polynomial in \( F[\bar{z}] \) obtained by substituting \( F_i(\bar{z}) \) for \( y_i \), \( i < m \), in \( g(\bar{y}) \). Since \( F \) is extremal, there exists a tuple of elements \( c_{ij} \in F, i < m, j < n, \) such that

\[ v_R G(\bar{c}) = \max\{ v_R G(\bar{d}) \mid d_{ij} \in F \}. \]

Putting \( e_j = \sum_{i<m} c_{ij} b_i \in F_0, j < n, \) we verify that

\[ v_{R_0} f(\bar{e}) = \max\{ v_{R_0} f(\bar{e'}) \mid \bar{e'} \in F_0 \}. \]

Indeed, let \( \bar{e'} \in F_0, e'_{ij} = \sum_{i<m} d_{ij} b_i, d_{ij} \in F; \) then \( v_{R_0} f(\bar{e'}) = m^{-1} v_R N_{F_0/F} f(\bar{e'}) = m^{-1} v_R G(\bar{d}) \leq m^{-1} v_R G(\bar{c}) = m^{-1} v_R G(\bar{d}) \leq m^{-1} v_{R_0} G(\bar{e}) = m^{-1} v_{R_0} N_{F_0/F} G(\bar{e}) = v_{R_0} f(\bar{e}). \)

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COROLLARY. Every extremal valued field is algebraically complete.

In fact, if every finite extension of a valued field is algebraically maximal then it is algebraically complete (cf. [1]). In what follows we argue to show that there exist algebraically complete valued fields that are not extremal.

Which valued fields are extremal? A question remains open whether maximal valued fields have this property. Below we come up with a positive result, which we think of as important in discussing issues dealing in elementary theory for formal power series fields over finite fields (cf. [3]).

Proposition 2. Let \( F = (F, R) \) be an algebraically complete discretely valued field, that is, \( \Gamma_R \simeq Z \). Then \( F \) is extremal.

Let \( f(x) \in F[x] \); without loss of generality (multiplying \( f \) by a suitable non-zero element of \( R \)), we may assume that \( f(\bar{x}) \in \bar{R}[\bar{x}] \). Proposition 3.1.7 in [1] shows that the following two cases are possible.

1. There exists a tuple \( \bar{a} \in R \) such that \( f(\bar{a}) = 0 \). Then \( v_R f(\bar{a}) = \omega = \max \{ v_R f(\bar{b}) \mid \bar{b} \in F \} \).

2. There exists \( \gamma_0 \in \Gamma_R \) such that \( v_R f(\bar{b}) \leq \gamma_0 \) for all \( \bar{b} \in F \). Let \( \bar{b}_0 \in F \) be an arbitrary tuple and \( \gamma_0 = v_R f(\bar{b}_0) \). Since \( \Gamma_R \simeq Z \), the interval \( [\gamma_0, \gamma_0] \) (in \( \Gamma_R \)) is finite and the set \( \{ v_R f(\bar{b}) \mid \bar{b} \in F \} \) is finite and non-empty. If \( \gamma_1 \) is the greatest element of that set, \( \bar{a} \in F \), and \( \gamma_1 = v_R f(\bar{a}) \), then it is obvious that

\[
v_R f(\bar{a}) = \max \{ v_R f(\bar{b}) \mid \bar{b} \in F \}.
\]

Remark 1. The property of being extremal is elementary, that is, it can be expressed via a (infinite) system of sentences in the language of the valuation theory of fields. This implies that every valued field, elementary equivalent to any field of formal power series (in one variable), is extremal.

Remark 2. The basic result (Thm. 1) of [4] follows immediately from Proposition 2.

We point out an important algebraic consequence of extremality, which fails for algebraically complete valuations. Let \( \langle D, R \rangle \) be a valued skew field (i.e., \( R \) is the valuation ring of a skew field \( D \); see [5]) and \( D_0 \subseteq D \) be a skew subfield of finite codimension. Then \( R_0 = R \cap D_0 \) is a valuation ring of the skew field \( D_0 \), the ramification index \( f = \abs{\Gamma_R : \Gamma_{R_0}} \) and the relative degree \( e = \abs{D : D_0} \) are finite, and \( e \cdot f \leq \abs{D : D_0} \); here, \( \Gamma_R (\Gamma_{R_0}) \) is a value group of the valuation ring \( R (R_0) \) and \( \bar{D} (\bar{D}_0) \) is a residue skew field of \( D (D_0) \). We say that an extension \( D \supseteq D_0 \) is defect free if \( \abs{D : D_0} = e \cdot f \).

THEOREM. Let \( F = (F, R) \) be an extremal valued field. Then every finite-dimensional skew field \( D \) with center \( F \) is a defect-free extension of \( F \).

If \( D \) is a finite-dimensional skew field over \( F \) then the property of \( R \) being Henselian implies that there exists a unique valuation ring \( R_0 \) of \( D \) such that \( R_0 \cap F = R \). And the fact that \( D \) is defect free over \( F \) means that \( n = e \cdot f \), where \( n = \abs{D : F}, e = \abs{\Gamma_{R_0} : \Gamma_R}, f = \abs{D : F}, \) and \( \bar{D} \) is the residue skew field of a ring \( \bar{R}_0 \) and \( \bar{F} \) is the residue field of a ring \( \bar{R} \).

MAIN LEMMA. Let \( F \leq E \leq D \) be a proper skew subfield. Then \( \bar{D} \neq \bar{E} \) or \( \Gamma_{R_1} \neq \Gamma_{R_0} \), where \( R_1 = R_0 \cap E \) is a valuation ring of the skew field \( E \) and \( \bar{E} \) is the residue skew field of a ring \( \bar{R}_1 \).

Let \( m = \abs{E : F} \). Choose a basis \( \varepsilon_i, i < m, \) for the skew field \( E \) over \( F \). Let \( d \in D \setminus E \). Extend a family \( \varepsilon_i, i < m, d, \) to a basis \( \delta_j, j < n, \) for the skew field \( D \) over \( F \) (\( \delta_i = \varepsilon_i, i < m, \delta_m = d \)).

Let \( g(x_0, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots, x_{n-1}) \) be the form (homogenous polynomial) over \( F \) corresponding to a reduced norm \( SN_{D/F} \) (cf. [5, VIII, 12.3]) for the basis \( \delta_j, j < n \). Let \( f(x_0, \ldots, x_{m-1}) = g(x_0, \ldots, x_{m-1}, -1, 0, \ldots, 0) \in F[x_0, \ldots, x_{m-1}] \). Since \( F \) is extremal, there exists a tuple \( a_0, \ldots, a_{m-1} \in F \) for which \( v_R f(\bar{a}) = \max \{ v_R f(\bar{b}) \mid \bar{b} \in F \} \). Notice that for every tuple \( \bar{b} \in F \), \( v_{R_0}(\varepsilon - d) = k^{-1} v_R f(\bar{b}) \),