GENERALIZED CONTRACTION MAPPING PRINCIPLES IN PROBABILISTIC METRIC SPACES

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Abstract. We give an improvement of Theorem 1 from [2] with a quite different approach, which enable us to prove that the fixed point is also globally attractive. In Theorem 2.11 a further generalization is obtained for a complete Menger space \((S, \mathcal{F}, T)\), where \(T\) belongs to a more general class of continuous \(t\)-norms than in the previous case where \(T = T_{M} (= \min)\). Theorem 3.2 is a generalization of Theorem 2 from [2]. Thereafter the notion of a generalized \(C\)-contraction of Krasnoselski’s type is introduced and a fixed point theorem for such mappings is proved. An application in the space \(S(\Omega, \mathcal{K}, P)\) is given.

1. Introduction

K. Menger introduced in [16] the notion of a probabilistic metric space. In Menger’s theory the concept of distance \(d(p, q)\) between two points \(p\) and \(q\) was considered as probabilistic, i.e., he proposed to replace the non-negative number \(d(p, q)\) by a distribution function \(F_{p,q} : \mathbb{R} \to [0, 1]\). Then for any real number \(x\) the value \(F_{p,q}(x)\) was interpreted as the probability that the distance between \(p\) and \(q\) is less than \(x\). The important development of the theory and applications of probabilistic metric spaces was due to B. Schweizer, A. Sklar and their collaborators (see the excellent monograph [26]).

In the last three decades many authors investigated the fixed point theory in probabilistic metric spaces (see the large bibliographies in [1, 6]). Some recent interesting applications of the fixed point theory in probabilistic metric spaces to the theory of random fractals were obtained in [12].

Let \(\mathcal{D}^+\) be the set of all distribution functions \(F\) such that \(F(0) = 0\) \((F\) is a nondecreasing, left continuous mapping from \(\mathbb{R}\) into \([0,1]\) such that \(\sup F(x) = 1\). The ordered pair \((S, \mathcal{F})\) is said to be a probabilistic metric space.
space if $S$ is a nonempty set and $\mathcal{F}: S \times S \to \mathcal{D}^+$ ($\mathcal{F}(p, q)$ is denoted by $F_{p,q}$ for every $(p, q) \in S \times S$) satisfies the following conditions:

1. $F_{u,v}(x) = 1$ for every $x > 0 \leftrightarrow u = v$ ($u, v \in S$).
2. $F_{u,v} = F_{v,u}$ for every $u, v \in S$.
3. $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x+y) = 1$ for $u, v, w \in S$ and $x, y \in \mathbb{R}^+$. If only 1 and 2 hold, the ordered pair $(S, \mathcal{F})$ is said to be a probabilistic semimetric space. A Menger space (see [24]) is a triple $(S, \mathcal{F}, T)$, where $(S, \mathcal{F})$ is a probabilistic metric space, $T$ is a triangular norm (abbreviated $t$-norm) and the following inequality holds:

$$F_{u,v}(x + y) \geq T(F_{u,w}(x), F_{w,v}(y))$$

for every $u, v, w \in S$ and every $x > 0$, $y > 0$.

Recall that a mapping $T: [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (a $t$-norm) if the following conditions are satisfied:

$$T(a,1) = a \text{ for every } a \in [0,1]; \quad T(a,b) = T(b,a) \text{ for every } a, b \in [0,1];$$

$$a \geq b, \ c \geq d \Rightarrow T(a,c) \geq T(b,d) \quad \left( a, b, c, d \in [0,1] \right);$$

$$T(a, T(b,c)) = T(T(a,b), c) \quad \left( a, b, c \in [0,1] \right).$$

**Example 1.1.** The following are the four basic $t$-norms:

(i) The minimum $t$-norm, $T_M$, is defined by

$$T_M(x,y) = \min(x,y),$$

(ii) The product $t$-norm, $T_P$, is defined by

$$T_P(x,y) = x \cdot y,$$

(iii) The Lukasiewicz $t$-norm $T_L$ is defined by

$$T_L(x,y) = \max(x + y - 1, 0),$$

(iv) The weakest $t$-norm, the drastic product $T_D$, is defined by

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

As regards the pointwise ordering, we have the inequalities

$$T_D < T_L < T_P < T_M.$$