ON EIGENVECTOR BOUNDS*

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Abstract.

We show under very general assumptions that error bounds for an individual eigenvector of a matrix can be computed if and only if the geometric multiplicity of the corresponding eigenvalue is one. Basically, this is true if not computing exactly like in computer algebra methods. We first show, under general assumptions, that nontrivial error bounds are not possible in case of geometric multiplicity greater than one. This result is also extended to symmetric, Hermitian and, more general, to normal matrices. Then we present an algorithm for the computation of error bounds for the (up to normalization) unique eigenvector in case of geometric multiplicity one. The effectiveness is demonstrated by numerical examples.


Key words: eigenvector inclusion, multiple eigenvalue, nonderogatory matrix.

1 Motivation.

The eigenproblem for a simple eigenvalue of a real or complex matrix can be directly transformed into a system of nonlinear equations. Based on that well known fact, methods are available to compute error bounds for the eigenvalue and corresponding normalized eigenvector. For an overview see, e.g., [7].

There are also Newton-like methods for multiple eigenvalues [2]; moreover, error bounds for double eigenvalues and the corresponding invariant subspace have been discussed in [1], and for multiple eigenvalues in [11]. The latter methods have in common that error bounds for base vectors of an invariant subspace are computed, not for an individual eigenvector. So the question may arise: Is it possible to compute (narrow) error bounds for an individual eigenvector to a multiple or nearly multiple eigenvalue? Exactly this question was posed to the first author [8] and lead to this note.

First, we note that the task can be accomplished using exact computations like in computer algebra methods. Every floating point number is a rational number, so a matrix $A$ with floating point entries is a matrix over the field of rational numbers. An eigenvalue of $A$ is a root of its characteristic polynomial,

the coefficients of which are sums of minors of $A$, and are therefore rational numbers. That means, every eigenvalue $\lambda$ of $A$ is an element of an algebraic extension field of the rationals. Basic operations in that field are effectively (and exactly) executable on the computer using well known methods from computer algebra [13], and any nontrivial vector in the nullspace of $A - \lambda I$ is an eigenvector to $\lambda$. Such a vector can, for example, be computed by Gaussian elimination with pivoting applied to $A^T - \lambda I$.

This approach allows to calculate eigenvectors (and eigenvalues) to arbitrary precision, and therefore also to calculate error bounds for an individual eigenvector. Note that for dimension greater than four Abel’s famous theorem tells, however, that the involved quantities can in general not be expressed in finite terms of roots and powers of rationals.

For larger dimensions this approach is hardly applicable since the involved quantities grow in length (i.e., number of digits) rapidly, and operations become expensive in terms of computing time. So the more practicable part of the question above is whether bounds for an individual eigenvector can be computed using floating point arithmetic.

We mentioned before that effective methods are available to compute (in floating point arithmetic) bounds for base vectors of an invariant subspace to a multiple or nearly multiple eigenvalue. Those methods have no restriction with respect to the Jordan structure, but they do not provide an inclusion of an individual eigenvector.

In this note we first show under general assumptions that a narrow inclusion of an individual eigenvector is not possible in case of a corresponding eigenvalue with geometric multiplicity greater than one.

One may feel uncomfortable with this statement for the following reason: Let $\lambda$ be an eigenvalue of algebraic multiplicity $m$. By the methods mentioned before [11] we may find inclusions $X_\nu \subseteq \mathbb{R}^n$ for base vectors of an invariant subspace of $A$. If the geometric multiplicity is assumed to be equal to the algebraic multiplicity, every vector $\sum \alpha_\nu X_\nu$ is an eigenvector inclusion, so especially $X_1$.

In such an argument an inherent discontinuity has been removed by assuming the geometric multiplicity to be equal to $m$. This is because a geometric multiplicity greater than one of an eigenvalue depends discontinuously on the matrix entries. The exact geometric multiplicity can only be computed if no rounding error whatsoever occurs, i.e., if all computations are exact. On the other hand, the property of an eigenvalue that its geometric multiplicity is one depends continuously on the matrix entries, i.e., it remains constant. This is only true for geometric multiplicity one.

The proofs are based on the construction of matrices with special properties in the neighborhood of $A$. These neighboring matrices are in general no longer symmetric or Hermitian when $A$ is. We show that even if attention is restricted to symmetric or Hermitian matrices, a narrow inclusion of an individual eigenvector is also not possible.

Again, one may feel uncomfortable about this statement because for a symmetric matrix it is known a priori that geometric and algebraic multiplicity