THE CLASS OF POLYHEDRAL COHERENT RISK MEASURES

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UDC 519.8

The class of polyhedral coherent risk measures that are used in making decisions under uncertainty is investigated. Operations are introduced on the measures of this class, and properties of these measures are studied. The problems of portfolio optimization based on the profitability–risk ratio for such risk measures are proved to be reducible to the corresponding linear programming problems.

Keywords: coherent risk measure, polyhedral coherent risk measure, conditional value-at-risk (CVaR), stochastic domination of the second order, optimal portfolio problem.

INTRODUCTION

This article is devoted to the investigation of properties of coherent risk measures and problems of searching for optimal solutions in portfolio problems that are connected with these measures. The presentation of its contents is restricted to the consideration of discretely distributed (on a finite set of elementary scenarios–events) random variables since this allows us to simplify the presentation and also is sufficient for modeling in financial applications. Note that the results presented can be extended to more general spaces after refinement of definite technical details.

During the development of the theory and applications in making decisions under uncertainty, various functions such as dispersion, semi-deviation, VaR (value-at-risk), and others, (see, for example, [1–4]) were used as risk measures. Then the axioms were formulated in [5] that must be satisfied by such a function called a coherent risk measure. Theoretically good properties provided by these axioms allow one to consider functions from this class as corresponding risk measures. In particular, such a measure is CVaR (conditional VaR) [6, 7] that is gradually introduced at the present time instead of the measure VaR into numerous financial applications. In some works (for example, in [8]), CVaR is called ES (expected shortfall). A definite generalization of CVaR is the concept of a spectral coherent risk measure, which is proposed in [9] and is, in essence, a convex combination of measures $\text{CVaR}_\alpha$ with various values of the parameter $\alpha$. As a result of such a construction of a spectral measure, this measure preserves all important properties of CVaR.

In [10], the class of polyhedral coherent risk measures is introduced, which is an important subset of the class of coherent measures. It contains all the well-known coherent risk measures, in particular, spectral measures and, moreover, guarantees the possibility of reducing portfolio optimization problems (maximization of profitability under constraints on the risk measure being used and minimization of the risk measure under constraints on profitability) to linear programming problems. In this article, the study of properties of this class (originated in [10]) of risk measures is continued. An interpretation of a coherent risk measure is given, operations on such measures are introduced, the conditions of their consistency with stochastic domination of the second order are investigated, and portfolio optimization problems with polyhedral coherent risk measures are considered.
DEFINITIONS AND SOME EXAMPLES

We consider only finite distributions of random variables whose observation is based on some number \( n \) of scenarios. Then to each scenario \( i = 1, \ldots, n \) corresponds a definite probability \( p_i > 0 \), i.e., some vector of probabilities \( p_0 = (p_0^0, \ldots, p_0^n) \). For each \( i = 1, \ldots, n \), \( p_i^0 > 0 \), it is given, a random quantity \( X \) is characterized by its distribution \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), and, hence, it is identified with an \( n \)-dimensional vector.

Let us introduce the following notations: \( \mathbb{I} = (1, \ldots, 1) \) and \( \mathbb{O} = (0, \ldots, 0) \) are \( n \)-dimensional vectors that consist of unities and zeros, respectively, \( S^n = \{ x = (x_1, \ldots, x_n) \} : \sum \lambda_i x_i \leq 1, x_i \geq 0, i = 1, \ldots, n \} \) is a unit simplex, \( \text{co} M = \{ \sum \lambda_i x_i : \lambda_i \geq 0, \sum \lambda_i = 1, x_i \in M, i = 1, 2, \ldots \} \) is the convex envelope of a set \( M \), and \( \text{ri} M \) and \( \text{cl} M \) are, respectively, the relative interior and closure of the set \( M \). By the relation \( x_1 \geq x_2 \) for \( x_1, x_2 \in \mathbb{R}^n \) we understand the corresponding componentwise inequality.

We recall that, according to [5], a function \( \rho : \mathbb{R}^n \to \mathbb{R} \) is called a coherent risk measure if the following axioms are fulfilled:

1. \( \rho (x + c \mathbb{I}) = \rho (x) - c \) for \( c \in \mathbb{R} \);
2. \( \rho (\mathbb{O}) = 0 \), \( \rho (c x) = c \rho (x) \) (positive homogeneity);
3. \( \rho (x_1 + x_2) \leq \rho (x_1) + \rho (x_2) \) (subadditivity);
4. \( \rho (x_1) \leq \rho (x_2) \) if \( x_1 \geq x_2 \) (monotonicity).

In this case, as is well known from [5, 10], the function \( \rho (\cdot) \) is of the form

\[
\rho (x) = \max \{ -x, p > / p \in P \},
\]

where \( P \) is some closed convex set of probability measures, i.e., we have \( P \subseteq S^n \). Since there exists a one-to-one correspondence between the function \( \rho (\cdot) \) and the set of measures \( P \), the specification of the set \( P \) by relation (1) actually specifies a coherent measure \( \rho (\cdot) \). In [10], this fact was used to define the class of polyhedral coherent measures. This name was assigned to functions of the form (1) for which the set \( P \) is representable in the form of a convex envelope consisting of a finite number of points. More precisely, if the set \( P \) is specified in the form

\[
P = \text{co} \{ p_i : i = 1, \ldots, k \},
\]

or in the equivalent form

\[
P = \{ p : Bp \leq c, p \geq 0 \},
\]

where \( B \) and \( c \) are a matrix and a vector (of corresponding dimensions) that depend on some parameters, then relations (1) and (2) uniquely specify a polyhedral coherent risk measure.

We first consider some examples of polyhedral coherent risk measures that are well known in financial problems. Let a distribution \( x = (x_1, \ldots, x_n) \) describe the profit made as a result of realization of scenarios with the corresponding probabilities \( p_0 = (p_0^0, \ldots, p_0^n) \).

**Example 1.** The worst-case risk (WCR) is the case of highest losses [11]. Then we have WCR \( (x) = \max \{ -x_i : i = 1, \ldots, n \} \), and the set \( P \) is of the form

\[
P_{\text{WCR}} = \{ p = (p_1, \ldots, p_n) : p_i \geq 0, i = 1, \ldots, n, \sum p_i = 1 \}.
\]

**Example 2.** The conditional value-at-risk (CVaR\(_\alpha\)) is the conditional mean of losses on the \( \alpha \)-tail of the distribution [6].

To avoid technical details, we do not cite the initial definition from [6] but use the following interpretation of this definition for discrete distributions:

\[
\text{CVaR}_\alpha (x) = \max \left\{ 1 / \alpha \left( \sum_{k_i = 1}^n p_i^0 \left( -x_{ik_i} + 1 \right) + \sum_{i = 1}^k p_i^0 (-x_{i}) \right) : \sum_1^{k_i} p_i^0 < \alpha \leq \sum_1^{k_i + 1} p_i^0 \right\}.
\]