Planar Problem of an Optimal Transfer of a Low-Thrust Spacecraft from High-Elliptic to Geosynchronous Orbit

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Abstract—Low-thrust flights from high-elliptic orbits are of considerable interest, since they allow one to decrease (compared to high-thrust flights) the propulsion consumption and to reduce the flight duration. At the same time, in comparison with the spiral unwinding flights from low near-circular orbits, this scheme minimizes the harmful effect of the radiation belts. Based on the maximum principle, the problem of optimization is reduced to a two-point boundary value problem, which is solved numerically using the modified Newton method. A method is suggested to obtain the initial approximation for solving the boundary value problem. The method takes advantage of the idea of transition from an approximately optimal trajectory to the optimal one. Two problems, which have different low-thrust models, are considered: one with permanently acting low thrust and the other with the possibility of turning it on/off. In both cases no restrictions are imposed on the thrust direction. A comparison of these problems is made. We investigated (i) what gain in the final mass can be attained when passing from the first to the second problem, (ii) at the cost of what loss in flight duration this can be achieved, and (iii) what changes in the optimal program of control must be done in this case.

INTRODUCTION

Recently a certain interest has been aroused in the possibility of using the low thrust for spacecraft injection into the geosynchronous orbit [4–7, 9–11]. In 1960s, for this aim (the transfer to a circular orbit and the boost up to the parabolic velocity of escape from the sphere of action of the Earth) much attention was given to studying the spiral unwinding from low circular or near-circular orbits, while at present the flights with low thrust from high-elliptic orbits are of great interest. The main advantages of a combined maneuver including (i) the transfer from a low orbit to a high-elliptic orbit using a large thrust and (ii) the subsequent flight to the geosynchronous orbit using a small thrust are as follows. The flight duration is smaller as well as the harmful effect on the serviceability of solar batteries (which are considered as a principal source of power supply onboard spacecraft) in comparison with purely spiral unwinding from a low orbit using a low thrust. At the same time, the propellant consumption is smaller as compared to a purely impulsive maneuver with a high thrust.

The large number of orbits around the Earth is a specific feature of trajectories under consideration: from several tens to a few hundreds (even up to one thousand). Accordingly, it is a considerable challenge to find the optimal control numerically in this case. For example, in the problem with a possibility to switch thrust on/off, one needs to determine zeros of the switch function with a sufficient accuracy, when their number can be as large as one–two thousand and more.

In the case of permanently acting thrust, experience showed the methods of averaging [3, 5–9] to work well. At another approach to problem’s solution, the structure of control is specified explicitly, and direct methods of optimization are used [1, 4]. One can also combine these approaches. Each of them has its characteristic advantages and disadvantages. For example, the direct methods of optimization turn out to be crude: the error of solving the optimization problem approximately is comparable to the influence of some problem parameters. In this case, the large random component does not allow one to obtain the result acceptable both quantitatively and qualitatively. The exact solution of the optimization problem presents especially severe difficulties [10].

In this paper we solve the optimization problem "exactly": based on the maximum principle a two-point boundary value problem is formulated, which is solved numerically. This allowed us to investigate some fine effects accurately. The problem is solved for two models of low thrust: (i) permanently acting and (ii) with a possibility to be switched on/off.

FORMULATION OF THE PROBLEM

The problem of an optimal flight of a low-thrust spacecraft is considered in the Earth’s central field. The optimal program of controlling the low-thrust vector
whose direction can be arbitrary is determined from the
maximum principle condition. It is found as a result of
solving a two-point boundary value problem. The
spacecraft’s final mass is the functional to be mini-
mized. The initial and final orbits are specified (Fig. 1).
The problem is solved in two different statements.

A. \( f \in \{0, f_{\text{max}}\} \), i.e., the thrust \( f \) can assume two val-
ues, zero and maximum. Accordingly, the trajectory is
composed of passive and active segments. The instants
of switching the thrust on and off are determined from
the condition of maximum Hamiltonian of the problem.
The flight duration \( T \) is specified. For the sake of sim-
plicity, the angular distance of a flight (reckoned from
the perigee of the elliptic orbit and measured in orbit
numbers \( N \)) will be specified by an integer numeral and
optimized.

B. \( f \in \{f_{\text{max}}\} \) is the permanently acting thrust. In this
case, the maximization problem for the final mass turns
out to be the minimization problem for the flight dura-
tion, i.e., the problem of minimal flight duration. Two
modifications of this problem are considered.

B1. The problem with a preset angular distance \( N \in \mathbb{R} \) (\( N \) can be noninteger).

B2. The problem with free \( N \in \mathbb{R} \) whose optimal
value should be found by solving the boundary value
problem.

BOUNDARY VALUE PROBLEMS

The spacecraft’s optimal motion in the planar case is
described by a system of differential equations of order
10 (see Appendix 1) for the phase variables \( \mathbf{x} = \langle m, h, \varphi, y, z \rangle \) and conjugate variables \( \lambda = \langle \lambda_m, \lambda_h, \lambda_\varphi, \lambda_y, \lambda_z \rangle \).

A. The initial \( t_0 \) and final \( t_G \) instants are specified.

At the instant \( t_0 \) (in the elliptical orbit) we have

\[
\begin{align*}
m(t_0) &= m_0, \quad h(t_0) = h_0, \quad \varphi(t_0) = \varphi_0, \\
y(t_0) &= y_0, \quad z(t_0) = z_0,
\end{align*}
\]

while the quantities

\[
\lambda_{m}(t_0), \lambda_{h}(t_0), \lambda_{\varphi}(t_0), \lambda_{y}(t_0), \lambda_{z}(t_0)
\]

are arbitrary.

At the instant \( t_G \) (in the geosynchronous orbit) we should have

\[
\begin{align*}
h(t_G) &= h_G, \quad \varphi(t_G) = \varphi_G, \\
y(t_G) &= y_G, \quad z(t_G) = z_G,
\end{align*}
\]

and the quantities

\[
\lambda_{h}(t_G), \lambda_{\varphi}(t_G), \lambda_{y}(t_G), \lambda_{z}(t_G)
\]

are arbitrary. The mass \( m(t_G) \) should be maximal, and it
is necessary for this that

\[
\lambda_{m}(t_G) \geq 0.
\]

The trajectory satisfying all the above requirements can
be solved in different ways.

A1. Since the conjugate variables are defined to an
accuracy of a constant positive factor, one of five quan-
tities (5) can be fixed (for example, \( \lambda_m(t_0) \)), while
remaining four values

\[
\lambda_{h}(t_0), \lambda_{\varphi}(t_0), \lambda_{y}(t_0), \lambda_{z}(t_0)
\]

can be adjusted in such a way that four essential bound-
dary conditions (3) would be satisfied.

Note that \( \lambda_{m} \) does not appear in the right-hand sides
of differential equations; therefore, the quantity \( \lambda_{m}(t_0) \)
fluences the trajectory only through the switch function \( \Lambda \), i.e., through the instants of switching the thrust
on and off.

A2. One can assume that \( \lambda_{m}(t_G) = 1 \) and seek the five
quantities

\[
m(t_G), \lambda_{h}(t_G), \lambda_{\varphi}(t_G), \lambda_{y}(t_G), \lambda_{z}(t_G),
\]

which would ensure five conditions (1) to be met.
Although the dimensionality of this boundary value
problem is larger, it has some advantages over the prob-
lem A1. For example, if there is some solution, and it is
necessary to find another solution with different param-
eters of the elliptic orbit, then the initial solution can be
taken as the initial approximation for a new boundary
value problem.

B1. In the minimal flight duration problem the thrust
operates permanently; therefore, \( \lambda_{m}(t_0) \) has no effect on
the quantities \( h(t_G), \varphi(t_G), y(t_G) \), and \( z(t_G) \). In this case
one can fix, for example, \( \lambda_{h}(t_0) \). Four boundary value
conditions (3) can be met by adjusting the appropriate
values of

\[
t_G, \lambda_{h}(t_0), \lambda_{\varphi}(t_0), \lambda_{y}(t_0).
\]

Fig. 1. The initial and final orbits of a spacecraft.