Elliptic Problems on Stratified Manifolds
in Spaces with Asymptotics: II

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This paper is a continuation of [1], where the theory of Sobolev problems was constructed in
spaces with asymptotics corresponding to a pair \((M, X)\), where \(M\) is a smooth compact manifold
without boundary and \(X\) is the union of two transversally intersecting smooth compact mani-
folds without boundary. Here we generalize the results of [1] to the case in which the strati-
fied manifold \(X\) is the union of finitely many transversally intersecting smooth compact mani-
folds without boundary; in other words, we construct the general theory of Sobolev problems in
spaces with asymptotics in the case of an arbitrary stratified compact manifold without boundary
and with transversal intersections.

In a sense, this paper concludes the series of papers [1, 3] dealing with Sobolev problems in
spaces with asymptotics defined near various manifolds.

First, the theory of Sobolev problems in spaces with asymptotics can be used in the quantum-
mechanical short-range interaction theory for the analysis of the Schrödinger equation with zero-
radius potential. For example, a self-adjoint extension of the corresponding operator was con-
structed in [4, 5] on the basis of the solution of a Sobolev problem in spaces with asymptotics on
the stratified manifold formed by a pencil of three intersecting planes.

Second, the technique developed in this paper for constructing the resolvent can be used in the
study of classical Sobolev problems (e.g., see [6]), since these problems and Sobolev problems in
spaces with asymptotics are very similar (for details, see [1]). For example, the problem [1]

\[
(I - \Delta)u(x) \equiv f(x) \quad (\text{mod } X), \quad \hat{i}^* \hat{B}u(x) = 0,
\]

where the equation is valid in the space \(\mathbb{R}^3\) outside the submanifold \(X = \{0\}\) and \(\hat{B}\) is an invertible
pseudodifferential operator of order \(\text{ord } B < -1\), has a unique solution in a space with asymptotics if
\(f(x) \in H^s(\mathbb{R}^3), \ s \geq -3/2\), and has no solution in spaces with asymptotics for the other values of \(s\).
If \(s \in [-5/2, -3/2]\), then problem (1) becomes a classical Sobolev problem and has a solution
\(u(x) \in H^{s+2}(\mathbb{R}^3)\). We see that one can obtain a classical Sobolev problem from a Sobolev problem
in spaces with asymptotics and vice versa by varying the smoothness of the right-hand side of the
equation.

Consider one more example. Let \(X\) be the simplest stratified manifold in \(\mathbb{R}^6\) defined as
\(X = L_1 \cup L_2\), where \(L_1\) and \(L_2\) are two three-dimensional planes in \(\mathbb{R}^6\) intersecting only at the
origin. By \(x\) we denote the coordinates along the first plane, and by \(t\) we denote the coordinates
along the second plane.

Consider the auxiliary problem

\[
(1 - \Delta)u \equiv f(x, t) \quad (\text{mod } X), \quad \hat{i}^* \hat{B}u = 0, \quad \hat{i}^{*}_L \hat{B}u = 0.
\]

Here \(f(x, t) \in H^s(\mathbb{R}^6)\), \(\hat{B}\) is a second-order invertible pseudodifferential operator, and the \(\hat{i}^*_L, i = 1, 2\), are the restriction operators induced by the embedding \(i: L_i \to \mathbb{R}^3\).

First, consider the case \(s \geq -1\). We construct a solution of the resulting problem in a space
with asymptotics, namely, in the form

\[
u(x, t) = u_0(x, t) + e^{-r_t} \frac{u_1(x)}{r_t} + e^{-r_x} \frac{u_2(t)}{r_x},\]

where \(u_0(x, t)\) is a solution of the auxiliary problem (2), and \(u_1(x)\) and \(u_2(t)\) are functions satisfying
the corresponding boundary conditions.

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Let us find the right-hand side of the last expression does not contain terms with derivatives of the delta function. We solve problem (5) following [4]. Then we obtain
\[ i_L \hat{B}(1 - \Delta)^{-1} (c_1(x) \delta(t) + c_2(t) \delta(x)) = -i_L \hat{B}(1 - \Delta)^{-1} f(x, t), \quad i = 1, 2. \]

We have thereby obtained the system
\[ \Delta c_1(x) + T_1 c_2(t) = F_1(x), \quad T_2 c_1(x) + \Delta_2 c_2(t) = F_2(t), \]
where \( F_i(x) = -i_L \hat{B}(1 - \Delta)^{-1} f(x, t), \) \( i = 1, 2, \) and \( \Delta, i = 1, 2, \) are the pseudodifferential operators \( \Delta = i_L \hat{B}(1 - \Delta)^{-1} i_L. \) Here by \( i_{L,i} \) we denote the operators of multiplication by the corresponding delta functions. One can readily show that \( \text{ord} \, \Delta = -1, \) since the operator \( \hat{B} \) is invertible, it follows that so are the operators \( \Delta, i = 1, 2. \) In (5), by \( T_i, i = 1, 2, \) we denote the translates \( T_i = i_L \hat{B}(1 - \Delta)^{-1} i_{L,i}, \) \( i \neq j. \) Obviously, \( T_i : H^s(R^3) \to H^{s+1}(R^3) \). A problem similar to (5) was studied in detail in [4]. We solve problem (5) following [4]. Then we obtain
\[ (1 - \Delta_1^{-1} T_1 \Delta_2^{-1} T_2) u_1(x) = \Delta_1^{-1} (F_1(x) - T_1 \Delta_2^{-1} F_2(t)). \]

It was shown in [5] that \( \Delta_1^{-1} T_1 \Delta_2^{-1} T_2 \) is a Fourier integral operator\(^1\) of the type \( \Phi_{1,1}. \) Let \( \hat{B} \) satisfy the condition that the operator \( (1 - \Delta_1^{-1} T_1 \Delta_2^{-1} T_2) \) is invertible. Then the solution of problem (4) can be represented in the form
\[ c_1(x) = (1 - \Delta_1^{-1} T_1 \Delta_2^{-1} T_2)^{-1} \Delta_1^{-1} (F_1(x) - T_1 \Delta_2^{-1} F_2(t)), \]
\[ c_2(t) = \Delta_2^{-1} (F_2(t) - T_2 c_1(x)). \]

Since \( F_i \in H^{s+4-3/2}(R^3) = H^{s+5/2}(R^3), \) we have \( c_1(x), c_2(t) \in H^{s+3/2}(R^3), \) which completes the proof. In the case under consideration, we have shown that \( c_1(x) \in H^{1/2}(R^3), c_2(t) \in H^{1/2}(R^3), \) and the solution of the problem can be represented in the form (4). Let us show that the function \( u(x, t) \) admits the representation (3). Since
\[ (1 - \Delta) \left( u_0(x, t) + e^{-r_t} \frac{u_1(x)}{r_t} + e^{-r_x} \frac{u_2(t)}{r_x} \right) \]
\[ = (1 - \Delta) u_0(x, t) - \delta(t) u_1(x) - \delta(x) u_2(t) - \frac{e^{-r_t}}{r_t} \Delta u_1(x) - \frac{e^{-r_x}}{r_x} \Delta u_2(t), \]
we have
\[ (1 - \Delta) u_0(x, t) - \delta(t) u_1(x) - \delta(x) u_2(t) - \frac{e^{-r_t}}{r_t} \frac{\partial^2}{\partial x^2} u_1(x) - \frac{e^{-r_x}}{r_x} \frac{\partial^2}{\partial x^2} u_2(t) \]
\[ = f(x, t) + c_1(x) \delta(t) + c_2(t) \delta(x). \]
It follows from the last relation that \( u_1(x) = -c_1(x), u_2(t) = -c_2(t), \) and
\[ u_0(x, t) = (1 - \Delta)^{-1} \left( f(x, t) + \frac{e^{-r_t}}{r_t} \frac{\partial^2}{\partial x^2} u_1(x) + \frac{e^{-r_x}}{r_x} \frac{\partial^2}{\partial x^2} u_2(t) \right). \]

Obviously, \( \frac{e^{-r_t}}{r_t} \frac{\partial^2}{\partial x^2} u_1(x), \frac{e^{-r_x}}{r_x} \frac{\partial^2}{\partial x^2} u_2(t) \in H^{-3/2}(R^3); \) therefore, \( u_0(x, t) \in H^{1/2}(R^3). \)

\(^1\) Fourier integral operators of the type \( \Phi_{1,1} \) were studied in [7], where necessary and sufficient conditions for their ellipticity were stated.