On a Test Statistic for Linear Trend

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Abstract. Let \( \{W(s)\}_{s \geq 0} \) be a standard Wiener process. The supremum of the squared Euclidian norm \( |Y(t)|^2 \), of the \( \mathbb{R}^2 \)-valued process \( Y(t) = (\sqrt{1/\sqrt{t}}W(t), \sqrt{1/12}\int_0^t s \, dW(s) - \sqrt{3/\sqrt{t}}W(t)) \), \( t \in [a, 1] \), is the asymptotic, large sample distribution of a test statistic for a change point detection problem, of appearance of linear trend. We determine the asymptotic behavior \( \mathbf{P}\{\sup_{t \in [a, 1]} |Y(t)|^2 > u\} \) as \( u \to \infty \), of this statistic, for a fixed \( a \in (0, 1) \), and for a “moving” \( \alpha = \alpha(u) \downarrow 0 \) at a suitable rate as \( u \to \infty \). The statistical interest of our results lie in their use as approximate test levels.

Key words. change point detection, \( \chi^2 \)-process, extremes, Gaussian process, linear trend, Ornstein–Uhlenbeck process, test of linear trend

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1. Introduction

Let \( \{W(t)\}_{t \in \mathbb{R}} \) be a standard Wiener process, and define

\[
Y(t) = \left( \frac{W(t)}{\sqrt{t}}, \frac{\sqrt{1/12} \int_0^t s \, dW(s) - \sqrt{3/\sqrt{t}}W(t)}{\sqrt{1/12}} \right)
\]

\[
= \left( \frac{W(t)}{\sqrt{t}}, \frac{\sqrt{3/\sqrt{t}}W(t) - \sqrt{1/12} \int_0^t W(s) \, ds}{\sqrt{1/12}} \right). \tag{1}
\]

We study the asymptotic behavior of

\[
\mathbf{P}\left\{ \sup_{t \in [a, 1]} |Y(t)|^2 > u \right\}, \quad \text{as } u \to \infty,
\]
for a constant \( z \in (0, 1) \), as well as for a ‘‘moving’’ \( z = z(u) \downarrow 0 \) at a certain suitable rate as \( u \to \infty \).

Consider a change point detection problem of appearance of linear trend, where the null hypothesis \( H_0 \): \( X_i = e_i \), for \( i = 1, \ldots, n \) is tested against the alternative

\[
H_1 : X_i = \begin{cases} 
  a_0 + a_1 (i/n) + e_i, & \text{for } i = 1, \ldots, k, \\
  e_i, & \text{for } i = k+1, \ldots, n,
\end{cases}
\]

for some \( k \in \mathbb{N} \) and \( a_0, a_1 \in \mathbb{R} \). Here \( \{e_i\}_{i=1}^\infty \) is standardized discrete white noise. Under \( H_0 \), the test statistic

\[
\max_{1 \leq k \leq n} \frac{(\sum_{i=1}^k X_i)^2}{k} + \frac{(\sum_{i=1}^k ((i/n) - (k+1)/(2n)) X_i)^2}{(\sum_{i=1}^k ((i/n) - (k+1)/(2n))^2)} \rightarrow \sup_{t \in [z, 1]} |Y(t)|^2, \quad \text{as } n \to \infty,
\]

where \( \rightarrow \) denotes weak convergence. See Jarušková (2000). Hence the upper tail of the law of \( \sup_{t \in [z, 1]} |Y(t)|^2 \) becomes the asymptotic, large sample distribution.

In Section 2, for easy reference, we collect some facts from the literature on extreme value theory, that are required for the proofs of our results.

In Section 3, we state and prove our first main result, Theorem 1. This theorem has the following immediate consequence, for the upper tail of \( \sup_{t \in [z, 1]} |Y(t)|^2 \):

\[
\lim_{u \to \infty} \frac{e^{u/2}}{u} \mathbb{P} \left\{ \sup_{t \in [z, 1]} |Y(t)|^2 > u \right\} = - \ln(z), \quad \text{for } z \in (0, 1).
\]

(2)

In Section 4, we state and prove our second main result, Theorem 2. This theorem has the following immediate consequence:

\[
\lim_{u \to \infty} \mathbb{P} \left\{ \sup_{t \in [\exp(-e^{u^2}/u), 1]} |Y(t)|^2 > u + 2x \right\} = 1 - \exp\{-e^{-x}\}, \quad \text{for } x \in \mathbb{R}.
\]

(3)

The literature on extremes of the norm of vector-valued Gaussian includes, for example, Sharpe (1978), Lindgren (1980, 1989), Albin (1990, Section 4; 2000, Section 5), and Piterbarg (1994). However, our component processes \( Y_1 \) and \( Y_2 \) are dependent, while the literature on non-differentiable processes only deal with independent components, and thus do not apply to \( Y \).

On the other hand, it is possible to derive our Theorem 1 from the literature on extremes of Gaussian fields, see Remark 1 below. However, this does not help for the proof of our Theorem 2, since that requires the proof of Theorem 1 anyway (see Section 2). In other words, an alternative, but longer way to our results, would be to use Gaussian field theory.