Generalized OWA Aggregation Operators

RONALD R. YAGER yager@panix.com
Machine Intelligence Institute, Iona College, New Rochelle, NY 10801, USA

Abstract. We extend the ordered weighted averaging (OWA) operator to provide a new class of operators called the generalized OWA (GOWA) operators. These operators add to the OWA operator an additional parameter controlling the power to which the argument values are raised. We look at some special cases of these operators. One important case corresponds to the generalized mean and another special case is the ordered weighted geometric operator.

Keywords: aggregation, generalized mean, fuzzy sets, OWA operators

1. Introduction

The ordered weighted averaging operator introduced in Yager (1988) provides a parameterized family of aggregation operators which have been used in many applications (Yager and Kacprzyk (1997)). In this work we provide a generalization of this OWA operator by combining it with the generalized mean operator (Dyckhoff and Pedrycz (1984)). This combination leads to a class of operators which we denote as the generalized ordered weighted averaging (GOWA) operators. Here we investigate some properties of these new operators.

2. GOWA Operators

The OWA operator is defined by

\[ F(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \]

where \( b_j \) is the \( j \)th largest of the \( a_i \) and \( w_j \) are a collection of weights such that \( w_j \in [0,1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

A convenient vector expression of this aggregation operator can be obtained if we let \( W \) be an \( n \)-dimension vector whose components are the \( w_j \) and let \( B \) be an \( n \)-dimension vector whose components are the \( b_j \). We call \( W \) the weighting vector and \( B \) the ordered argument vector. Using these vectors we can express \( F(a_1, \ldots, a_n) = W^T B \).

By selecting different manifestations of \( W \) we can implement different aggregations. Particularly notable among the operators that can be obtained are the Max,
Min and the simple average. These are respectively obtained by the vectors $W^*$ where $w_1 = 1$ and $w_j = 0$ for $j \neq 1$, $W$, where $w_n = 1$ and $w_j = 0$ for $j \neq n$, and $W_A$ where $w_j = \frac{1}{n}$. Yager (1993) discusses various different examples of weighting vectors.

It has been shown (Yager (1988)) that the OWA operator is a mean operator: it is symmetric, monotonic and bounded, $\text{Min}[a_i] \leq F(a_1, \ldots, a_n) \leq \text{Max}[a_i]$. It is also idempotent, $F(a_1, \ldots, a_n) = a$ when $a_i = a$ for all $i$.

While the OWA operator can take its arguments values from the real line an important special case occurs when the arguments are drawn from the unit interval, $I = [0, 1]$. In this case $F: \mathbb{F} \to I$. It is this special case we shall focus on.

We now introduce a class of aggregation operator which we shall call the generalized OWA operators. We shall denote these as GOWA operators.

**Definition**

A mapping $M : \mathbb{F}^n \to I$ is called a generalized ordered weighted aggregation (GOWA) operator of dimension $n$ if

$$M(a_1, \ldots, a_n) = \left( \sum_{j=1}^n w_j b_j^\lambda \right)^{1/\lambda}$$

where, $w_j$ is a collection of weights satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$; $\lambda$ is a parameter such that $\lambda \in [-\infty, \infty]$; $b_j$ is the $j$th largest of the $a_i$.

Using vector notation we can express this as $M(a_1, \ldots, a_n) = (W^TB^\lambda)^{1/\lambda}$ where $W$ and $B$ are the vectors introduced earlier. In order to emphasize the parameters $W$ and $\lambda$ at times we shall indicate this operator as $M_{W,\lambda}(a_1, \ldots, a_n)$.

Two special cases are of great significance. First is the case when $\lambda = 1$, here we get

$$M(a_1, \ldots, a_n) = \sum_{j=1}^n w_j b_j = W^T B$$

which is the usual OWA operator. The other important special case is when $w_j = \frac{1}{n}$. In this case

$$M(a_1, \ldots, a_n) = \left( \sum_{j=1}^n \frac{1}{n} a_j \right)^{1/\lambda}$$

This is the generalized mean operator discussed by Dyckhoff and Pedrycz (1984). We note these are also mean operators: they are symmetric, monotonic and bounded.

Before investigating more special cases we look at some properties of the GOWA operators. First we see that the GOWA operator is commutative, if $\Pi$ is any permutation then

$$M(a_1, \ldots, a_n) = M(a_{\Pi(1)}, \ldots, a_{\Pi(n)})$$

This implies that the initial indexing of the arguments does not matter.