On augmented Lagrangians for Optimization Problems with a Single Constraint

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Abstract. We examine augmented Lagrangians for optimization problems with a single (either inequality or equality) constraint. We establish some links between augmented Lagrangians and Lagrange-type functions and propose a new kind of Lagrange-type functions for a problem with a single inequality constraint. Finally, we discuss a supergradient algorithm for calculating optimal values of dual problems corresponding to some class of augmented Lagrangians.

Key words: Augmented Lagrangians, Lagrange-type functions, Supergradient method

1. Introduction

Classical Lagrange and penalty functions can be applied only for examination of some special classes of constrained optimization problems. Some generalizations of these functions have recently been studied. Currently there are two main types of such a generalization. One of them is the augmented Lagrangian, which is based on an augmentation of the classical Lagrange function by a certain augmenting function (see [6, 12] and references therein).

The fundamental of the other approach to generalization of Lagrangians is a nonlinear convolution of the objective and constraint functions (see [7, 9, 10, 12] and references therein). Such a convolution leads to nonlinear Lagrange-type functions. We establish some links between the two mentioned approaches.

It is well-known that each constrained optimization problem can be reformulated as a problem with a single inequality-constraint. Many complicated constructions become much simpler and more understandable for single-constrained problems. In this paper we examine the augmented Lagrangians with certain augmenting functions for problems with a single (either inequality or equality) constraint. In particular we study the so-called sharp Lagrangian [6] for such problems. The simple structure of the sharp augmenting function $\sigma(z) = |z|$ allows us to give an explicit description of the sharp augmented Lagrangian. By using this result we propose a new type of nonlinear Lagrangian for problems with an inequality constraint, for which the dual function can be easily expressed through the dual
function of the problem with an equality constraint. This approach allows us to extend some results, obtained for problems with an equality constraint, to problems with an inequality constraint. We also examine a certain version of the supergradient method for solving the dual problem. First we consider problems with an equality constraint and generalize a version of this method proposed in [1] for sharp augmented Lagrangian, to a more general class of augmented Lagrangians. Then we show that a new type of nonlinear Lagrangians allows to use this method in solving the problems with an inequality constraint.

2. Preliminaries

2.1. CONSTRAINED OPTIMIZATION PROBLEM AND ITS REFORMULATION

Consider a metric space $X$ and functions $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}^m$ where $\mathbb{R}^m$ is $m$-dimensional Euclidean space equipped with the coordinate-wise order relation. Consider the following constrained optimization problem

$$P(f, g): \minimize_{x \in X} f(x) \text{ subject to } g(x) \leq 0,$$

where $g(x) = (g_1(x), \ldots, g_m(x))$. Let

$$X_0 = \{x \in X : g(x) \leq 0\}$$

be the set of feasible solutions for $P(f, g)$ and let

$$M = \inf \{f(x) : x \in X_0\}$$

be the optimal value of $P(f, g)$. It is assumed that $M > -\infty$.

For some applications it is convenient to consider certain reformulations of the problem $P(f, g)$. In particular, $P(f, g)$ can be reformulated as the following problem $P(f_1, g)$ with the single constraint function $f_1$:

$$P(f_1, g): \minimize_{x \in X} f_1(x) \text{ subject to } g(x) \leq 0,$$

where $f_1(x) = \max_{i=1, \ldots, m} g_i(x)$. Problems $P(f, g)$ and $P(f_1, g)$ have the same set of feasible solutions and the same objective function.

REMARK 2.1. A general mathematical programming problem:

$$P(f, g, h): \minimize_{x \in X} f(x) \text{ subject to } g_i(x) \leq 0 (i \in I), \quad h_j(x) = 0 (j \in J)$$

with finite $I$ and $J$, also can be reformulated as (2.4) with

$$f_1(x) = \max_{i \in I} \max_{j \in J} |h_j(x)|.$$

The approach proposed here is suitable for examining even more complicated problems, which lead to $f_1(x) = \min_{i \in I} \max_{j \in J} g_{ij}(x)$, where $I$ and $J$ are finite sets of indices and $g_{ij}$ are certain functions.