TECHNICAL NOTE

Cutting Sphere Algorithm

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Abstract. We present a generalization of the conventional cutting plane algorithm for the solution of nonconvex optimization problems with nonsmooth inequality constraints. The cuts are effected using spheres rather than hyperplanes.

Key Words. Nonsmooth optimization algorithms, outer approximations, cutting plane algorithm.

1. Introduction

This note is a first attempt to construct an algorithm for solving a general class of nonsmooth optimization problems in the form

$$\min_{x \in \mathbb{R}^n} \{ f^0(x) \mid f(x) \leq 0, x \in X \},$$

where $x \in \mathbb{R}^n$, $f^0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth, $f : \mathbb{R}^n \rightarrow \mathbb{R}^q$ is continuous but is not known to have gradients or subgradients, and

$$X = \{ x \in \mathbb{R}^n \mid g(x) \leq 0 \},$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^q$. If not smooth, convert (P) into the equivalent form

$$\min_{x \in \mathbb{R}^n} \{ f^0(x) \leq x^0, f(x) \leq 0, x \in X \},$$

where $x = (x^0, x) \in \mathbb{R}^{n+1}$. Support from National Science Foundation Grant ECS-9900985, UC Berkeley Space Sciences Laboratory, and Lockheed Martin Advanced Technology Center Minigrant Program is acknowledged. Professor, Department of Electrical Engineering and Computer Science, University of California, Berkeley, California. Graduate Student, Department of Civil and Environmental Engineering, University of California, Berkeley, California. If not smooth, convert (P) into the equivalent form
with \( g: \mathbb{R}^n \to \mathbb{R}^e \) smooth. We use the notation \( u \leq 0 \) to denote \( u^i \leq 0, \ldots, u^l \leq 0 \), where \( u = (u^1, \ldots, u_q) \).

In particular, problems of the form (P) arise in design optimization of structures under uncertainty, where the probability of failure of a structure or a component enters as a constraint function (see e.g. Ref. 1). The probability of failure, as a function of the design variables \( x \), is given by

\[
p(x) = \int_{\Omega(x)} f_V(v) \, dv,
\]

where \( f_V \) is the joint probability density function for an \( m \)-dimensional vector of random variables \( V \) and

\[
\Omega(x) = \{ v \in \mathbb{R}^m \mid h(x, v) \leq 0 \}
\]

is the failure domain for the structure, with \( h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) smooth (Ref. 2). The probability of failure is known only to be Lipschitz continuous (Ref. 3), and hence a problem with a constraint of the type \( p(x) \leq 0 \) is of the form (1). Even if the probability of failure were differentiable, the absence of an efficient way of computing the gradients of \( p(x) \) makes it seemingly impossible to develop a gradient-based algorithm for problems with functions defined in terms of \( p(x) \).

In the next section, we will develop a conceptual algorithm for solving (P), which does not require \( f(\cdot) \) to be differentiable.

### 2. Conceptual Cutting Sphere Algorithm

We will develop a conceptual algorithm for (P), which is inspired by the classic cutting plane method (Refs. 4–5).

The cutting plane method is easiest to explain when applied to problems with a single inequality constraint, as shown below:

\[
\min_{x \in \mathbb{R}^n} \{ \tilde{f}^0(x) \mid \tilde{f}(x) \leq 0 \},
\]

where \( \tilde{f}^0(\cdot) \) and \( \tilde{f}(\cdot) \) are smooth, convex functions.

The cutting plane method constructs a polyhedral approximation to the constraint set \( \{ x \in \mathbb{R}^n \mid \tilde{f}(x) \leq 0 \} \), which becomes progressively more and more accurate in the vicinity of a solution, as follows. One begins with a given starting point \( x_0 \), and at iteration \( i \) one computes \( x_{i+1} \) as the solution of the problem

\[
\min_{x \in \mathbb{R}^n} \{ \tilde{f}^0(x) \mid \tilde{f}(x_k) + (\nabla \tilde{f}(x_k), x - x_k) \leq 0, k = 0, 1, 2, \ldots, i \}.
\]