UNIVERSAL BOUNDARY SETS IN THE GENERALIZED STEINER PROBLEM

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It is proved that every simple binary tree is realized by a nondegenerate minimal parametric network spanning the vertices of a regular simplex. Bibliography: 9 titles.

§1. Introduction

In the present paper, we consider the problem on a universal boundary set. This problem is among the questions integrated by the name the generalized Steiner problem. The generalized Steiner problem is as follows: to describe all locally minimal networks spanning a given boundary set \( A \). A. O. Ivanov and A. A. Tuzhilin [1] were first to formulate the problem in this form. They extensively investigated this problem for the case of the plane \( \mathbb{R}^2 \) in a series of papers. However, practically nothing is known in the case where \( n > 2 \), except a number of examples [2, 3].

We restrict ourselves to consideration of locally minimal networks without cycles. It is known that any such a network uniquely splits into locally minimal binary trees. If the boundary set is not fixed, then any binary tree is realized by a locally minimal planar network [1]. Generally speaking, this is not already so in the case of a fixed boundary set \( A \). A natural question arises [1]: does there exist a universal boundary set? I.e., a set \( A \) such that any binary tree \( G \) spanning \( A \) is realized by a locally minimal network with boundary \( A \).

In the case of the plane (\( n = 2 \)), the answer appears to be in the negative. For example, a generic four-point set \( A \) is spanned by at most two locally minimal networks, see [4]. At the same time, \( A \) is spanned by three binary trees. A generic five-point set \( A \) is spanned by at most eight locally minimal networks, though \( A \) is spanned by 15 binary trees.

Ivanov and Tuzhilin conjectured that if \( A \) is the set of the vertices of a regular \( n \)-simplex, then any binary tree spanning \( A \) is realized by a locally minimal network. It turns out that even a stronger result holds true (see §3).

Main theorem. Any simple (i.e., without vertices of degree 2) tree spanning the vertices of a regular \( n \)-simplex is realized by a nondegenerate minimal parametric network.

We recall basic definitions of the general theory of minimal networks (the most part of them is taken from [5, 6]).

2.1. Topological graphs. A topological graph \( G \) is an arbitrary finite one-dimensional CW-complex with a fixed cellular decomposition. The 0-cells are vertices, and 1-cells are edges of \( G \).

Remark. It is clear that we obtain a topological representation of abstract graphs of most general form, i.e., graphs with loops and multiple edges. For this reason, all terminology both of abstract graph theory and of set-theoretic topology is directly transferred to topological graphs. For the brevity, topological graphs are simply called graphs in what follows.

Let \( e \) be an edge of a graph \( G \). If we regard all points of \( e \) as equivalent, then we say that the factor-graph \( \tilde{G} = G/e \) is obtained from \( G \) by reduction of the edge \( e \).

Assume that \( M \) is a certain fixed subset of the vertex set of \( G \). Then the pair \((G, M)\) is a graph with boundary \( M \). Sometimes, we informally say that \( G \) has boundary \( M \) and denote \( M \) by \( \partial G \).

The vertices in \( \partial G \) are boundary or fixed, and the remaining vertices of \( G \) are inner or movable. The edges incident to boundary vertices are boundary, and the remaining edges of \( G \) are inner.
Let $G$ be an arbitrary graph with boundary $\partial G$ (maybe empty), and let $P \in G$ ($P$ is not necessarily a vertex of $G$, see the Remark above). An admissible neighborhood $U \subset G$ if $P$ is the closure of a connected neighborhood of $P$ containing no vertices of $G$ different from $P$ if $P$ is a vertex, and containing no loops in $G$.

We regard $U$ as a graph: the vertices of $U$ are all points in $\Delta U \cup \{P\}$, and the edges are the segments in $U$ joining the vertices. The obtained tree $G_U$ is the local graph of $G$ with center at $P$. By definition, the canonical boundary $\partial G_U$ of $G_U$ consists of all vertices in $\Delta U$, and also of the vertex $P$ if $P$ is a boundary vertex of $G$.

2.2. Parametric networks. Let $G$ be a connected graph. A piecewise smooth mapping $\Gamma: G \to \mathbb{R}^n$ is a (parametric) network, and $\Im \Gamma = \Gamma(G) \subset \mathbb{R}^2$ is the trace of $\Gamma$. The graph $G$ is the parametrizing graph of $\Gamma$, and the class of graphs isomorphic to $G$ is the topology of $\Gamma$.

The restriction of $\Gamma$ to a vertex of $G$ is a vertex, and the restriction to an edge is an edge of $\Gamma$. The length of $\Gamma$ is the sum of the lengths of the edges of $\Gamma$.

Below, we use the following notation: $\Gamma[G]$ is a network with parametrizing graph $G$, and $\Gamma[e]$ is the edge of $\Gamma[G]$ corresponding to the edge $e$ of $G$.

Let $\Gamma[G]$ be a parametric network, and let $\partial G$ be the boundary of $G$. The restriction $\Gamma|_{\partial G}$ is a subset $\partial \Gamma$ of the vertex set of $\Gamma$, which we call the boundary of $\Gamma$. If $\mathcal{A} \subset \mathbb{R}^n$ is a finite set and $\Gamma$ is a network such that the image of $\partial \Gamma$ coincides with $\mathcal{A}$, then we say that $\Gamma$ spans $\mathcal{A}$ along the boundary mapping $\partial \Gamma$. We also say that $\mathcal{A}$ is the boundary set for $\Gamma$.

The vertices of $\Gamma$ in $\partial \Gamma$ are boundary or fixed, and the remaining vertices of $\Gamma$ are inner or movable. The edges of $\Gamma$ that are incident to boundary vertices are boundary, and the remaining edges are inner.

Let $\Gamma[G]$ be an arbitrary network, let $P$ be any point of the graph $G$, and let $G_U$ be the local graph of $\Gamma[G]$ with center at $P$. The network $\Gamma_U$ with boundary $\Gamma|_{G_U}$ is the local network with center at $P$. The restriction $\Gamma|_{\partial G_U} = : \partial \Gamma_U$ is the canonical boundary of $\Gamma_U$.

2.3. Locally minimal networks. A network $\Gamma$ spanning a finite set $\mathcal{A} \subset \mathbb{R}^n$ is absolutely minimal if the length of $\Gamma$ is the least among the lengths of all networks spanning $\mathcal{A}$.

The network $\Gamma$ is locally minimal if for each point $P$ of $\Gamma$ a certain local network with center $P$ is absolutely minimal.

An edge $\Gamma[e]$ of the network $\Gamma[G]$ is degenerate if it has zero length. The corresponding edge $e$ of $G$ is also called degenerate. It is obvious that $\Gamma$ maps the entire degenerate edge $e$ into a point. Networks without degenerate edges are nondegenerate.

The following theorem completely describes the local structure of nondegenerate locally minimal networks (for the proof see, e.g., [7]).

**Theorem 2.1.** A nondegenerate network $\Gamma$ with boundary $\mathcal{A}$ is locally minimal if and only if the following conditions (1)–(4) are fulfilled:

1. the edges of $\Gamma$ are rectilinear segments,
2. the angle between any two adjacent edges is at least $120^\circ$,
3. all vertices of degree 1 are boundary,
4. if a vertex $v$ of degree 2 is not boundary, then the angle between the edges incident to $v$ is equal to $180^\circ$.

Conditions (1), (3), and (4) are typical for criteria of various notions of minimality of networks. They motivate initial restrictions put on the classes of networks considered, as well as the following definitions.

**Definition.** A network $\Gamma$ is linear if all edges of $\Gamma$ are rectilinear segments. We denote by $\mathcal{L}[G, \mathcal{A}, b]$ the set of all linear networks $\Gamma$ with fixed topology $G$, boundary set $\mathcal{A}$, and boundary mapping $b: \partial G \to \mathcal{A}$.

**Definition.** A connected topological graph $G$ is a simple tree if $G$ is a tree and has no vertices of degree 2. In what follows, all parametrizing graphs of the networks considered are assumed to be simple trees, if not indicated otherwise.

Condition (2) of Theorem 2.1 implies that each vertex of a nondegenerate locally minimal network has degree at most three. From simple trees, we single out an important class of graphs: binary trees (or 2-trees), where each vertex has degree 1 or 3.

For all simple trees, we assume that all vertices of degree 1 are boundary, and the remaining vertices are movable.