ABSTRACT. In this paper, the authors discuss Frege’s theory of “logical objects” (extensions, numbers, truth-values) and the recent attempts to rehabilitate it. We show that the ‘eta’ relation George Boolos deployed on Frege’s behalf is similar, if not identical, to the encoding mode of predication that underlies the theory of abstract objects. Whereas Boolos accepted unrestricted Comprehension for Properties and used the ‘eta’ relation to assert the existence of logical objects under certain highly restricted conditions, the theory of abstract objects uses unrestricted Comprehension for Logical Objects and banishes encoding (eta) formulas from Comprehension for Properties. The relative mathematical and philosophical strengths of the two theories are discussed. Along the way, new results in the theory of abstract objects are described, involving: (a) the theory of extensions, (b) the theory of directions and shapes, and (c) the theory of truth values.

KEY WORDS: abstract objects, extensions, George Boolos, Gottlob Frege, Hume’s Principle, logical objects, numbers, object theory, second-order logic, truth values

In 1884, Frege formulated some ‘abstraction’ principles that imply the existence of abstract objects in classical logic. The most well-known of these is:

Hume’s Principle:
The number of F’s is identical to the number of G’s iff there is a one-to-one correspondence between the F’s and the G’s.

\[
\#F = \#G \iff F \approx G
\]

When added to classical second-order logic (but not free second-order logic), this implies the existence of numbers, which Frege regarded as ‘logical objects’. He also developed analogous principles for such abstract objects as directions and shapes:

Directions:
The direction of line a is identical to the direction of line b iff a is parallel to b.

\[
\vec{a} = \vec{b} \iff a \parallel b
\]

Shapes:
The shape of figure a is identical to the shape of figure b iff a is (geometrically) similar to b.

\[
\widehat{a} = \widehat{b} \iff a \sim b
\]
With a system that allows for propositions (intensionally conceived) and distinguishes them from truth-values, one could also propose a principle governing truth-values in a way analogous to the above. Of course, Frege wouldn’t have formulated such a principle. In his system, sentences denote truth values and the identity symbol can do the job of the biconditional. Thus, in his system, the truth value of \( p \) equals the truth value of \( q \) just in case \( p = q \). But if we use a modern-day predicate logic instead of a term logic, distinguish propositions from truth values, and allow the propositional variables ‘\( p \)’ and ‘\( q \)’ to range over propositions, something like the following principle governing truth values would be assertible for a modern-day Fregean (Boolos, 1986, p. 148):

**Truth Values:**
The truth value of \( p \) is identical with the truth value of \( q \) iff \( p \) is equivalent to \( q \).

\[
(p^\circ = q^\circ) \leftrightarrow (p \leftrightarrow q)
\]

Here, and in what follows, the biconditional is *not* to be construed as an identity sign.

Frege might have called all of these objects ‘logical objects’, since in (Frege, 1884), he thought he had a way of defining them all in terms of a paradigm logical object, namely, extensions. Let us for the moment use \( \alpha, \beta \) as metavariables ranging over variables for objects, concepts, or propositions, \( \varphi \) as a metavariable ranging over formulas, and \( [\lambda \alpha \varphi] \) for the extension of the (first- or second-level) concept \( [\lambda \alpha \varphi] \). Frege (1884, §68) then defined ‘the number of \( F \)s’ (\( \#F \)), ‘the direction of line \( a \)’ (\( \vec{a} \)), and ‘the shape of figure \( c \)’ (\( \vec{c} \)) as follows:

\[
\#F =_{df} [\lambda G \, G \approx F]
\]

\[
\vec{a} =_{df} [\lambda x \, x \parallel a]
\]

\[
\vec{c} =_{df} [\lambda x \, x \sim c]
\]

If we ignore the infamous Section 10 of Frege’s *Grundgesetze*, then the above definitions suggest the following definition of ‘the truth value of proposition \( p \)’ (\( p^\circ \)), given the principle Truth Values:

\[
p^\circ =_{df} [\lambda q \, q \leftrightarrow p]
\]

All of these logical objects would thereby have been systematized by:

**Basic Law V:**
The extension of the concept \( [\lambda \alpha \varphi] \) is identical to the extension of the concept \( [\lambda \alpha \psi] \) iff all and only the objects falling under the concept \( \varphi \) fall under the concept \( \psi \).

\[
[\lambda \alpha \varphi] = [\lambda \alpha \psi] \iff \forall \beta ([\lambda \alpha \varphi] \beta \leftrightarrow [\lambda \alpha \psi] \beta)
\]