Numerical simulation of two-phase flow through heterogeneous porous media

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Received 4 December 2001; accepted 4 February 2003

A mixed finite element method is combined to finite volume schemes on structured and unstructured grids for the approximation of the solution of incompressible flow in heterogeneous porous media. A series of numerical examples demonstrates the effectiveness of the methodology for a coupled system which includes an elliptic equation and a nonlinear degenerate diffusion–convection equation arising in modeling of flow and transport in porous media.

Keywords: finite volume method, mixed hybrid finite element, nonlinear convection–diffusion, porous media, unstructured grids

AMS subject classification: 76M12, 65M12, 35K65

1. Introduction

Multiphase flow of fluids in porous media is physically and chemically complex. It involves heterogeneities in the porous media at many different length scales and complicated processes such as diffusion and dispersion. It is well known that the transport term in the flow equations are governed by fluid velocities. Thus accurate numerical simulation requires accurate approximations of these velocities. Often the flow properties of the porous media vary abruptly with sharp changes in lithology. Consequently, the coefficient in the flow equation is quite rough.

Flow simulation in petroleum and environmental applications has been extensively studied using finite element methods in the last two decades (see, e.g., [9, and references therein]). Also, discretizations using both finite element and finite volume methods are presented in [7]. More recently, finite volume methods were developed and analyzed for immiscible two-phase flow in porous media in the case where the diffusion term is neglected (see [8, and references therein]). This approach leads to robust schemes applicable for unstructured grids and the approximate solution has various interesting
properties which correspond to the properties of the physical solution. These methods have been useful for advective flow problems because they combine element by element conservation of mass with numerical stability and minimal numerical diffusion.

The purpose of this paper is to discuss the applicability of the mixed finite element methods and finite volume methods to various types of flow in porous media. These problems occur namely in saltwater intrusion, in groundwater contaminant transport, nuclear waste repository studies and petroleum reservoir simulation. We focus on two-phase immiscible flow in heterogeneous porous media. The equation governing these types of flow can be effectively rewritten in a fractional flow formulation; i.e., in terms of a global pressure and saturation as the primary variables; see, e.g., [6]. This formulation leads to a coupled system of partial differential equations which includes an elliptic pressure–velocity equation and a nonlinear degenerate parabolic saturation equation. The saturation equation is convection dominated and thus special care should be taken in discretization. The diffusion term is small but important and cannot be neglected.

The outline of the paper is as follows. Section 2 contains a short description of the mathematical and physical model used in this study. In section 3, the numerical schemes are presented with emphasis in section 3.1 on the mixed finite element method employed for the solution of the pressure equation. Then, the vertex-centered finite volume method used for the solution of the saturation equation is illustrated. Section 4 is devoted to the presentation of the results of some selected numerical investigations for the sake of illustration. Results are given for a quarter five-spots problem with variable permeability. Additional conclusions are drawn in section 5.

2. Mathematical and physical model

We consider the immiscible displacement of one incompressible fluid by another in a horizontal reservoir \( \Omega \subset \mathbb{R}^2 \) over a time interval \([0, \tau]\). The governing equations can be written as follows (see [6]):

**Pressure equation:**

\[
\begin{align*}
\bar{q} &= -d(u)K(x)\nabla P; \\
\bar{q} \cdot \vec{n}\big|_{\Gamma_1} &= -q_d; \\
\bar{q} \cdot \vec{n}\big|_{\Gamma_2} &= 0; \\
P|_{\Gamma_3} &= P_0 \text{ on } [0, \tau].
\end{align*}
\]

**Saturation equation:**

\[
\begin{align*}
0 \leq u(x, t) \leq 1, \\
\frac{\partial u}{\partial t} + \nabla (b(u)\bar{q}) - \nabla (K(x)\nabla \alpha(u)) &= 0 \text{ in } \Omega, \\
u|_{\Gamma_1} &= 1; \\
K\nabla \alpha(u) \cdot \vec{n}\big|_{\Gamma_2} &= 0; \\
u|_{\Gamma_3} &= 0 \text{ on } [0, \tau], \\
u(x, 0) &= u_0(x) \text{ in } \Omega,
\end{align*}
\]

where the boundary \( \Gamma \) splits up into three parts such that \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \) and \( \Gamma_i \cap \Gamma_j = \emptyset \) for \( i \neq j \); \( \Gamma_1 \) is the part of the boundary where the water is injected, \( \Gamma_2 \) is the impervious part of the boundary and \( \Gamma_3 \) is the producing part of the boundary. Here \( K(x) \) and \( \Phi(x) \) are the absolute permeability tensor and porosity of the porous