ALGEBRA WITH QUADRATIC COMMUTATION RELATIONS FOR AN AXIALLY PERTURBED COULOMB–DIRAC FIELD

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The motion of a particle in the field of an electromagnetic monopole (in the Coulomb–Dirac field) perturbed by an axially symmetric potential after quantum averaging is described by an integrable system. Its Hamiltonian can be written in terms of the generators of an algebra with quadratic commutation relations. We construct the irreducible representations of this algebra in terms of second-order differential operators; we also construct its hypergeometric coherent states. We use these states in the first-order approximation with respect to the perturbing field to obtain the integral representation of the eigenfunctions of the original problem in terms of solutions of the model Heun-type second-order ordinary differential equation and present the asymptotic approximation of the corresponding eigenvalues.

Keywords: integrable systems, Dirac monopole, nonlinear commutation relations, coherent states, asymptotic spectrum behavior

This paper opens a cycle of papers dedicated to the thirtieth anniversary of V. P. Maslov’s book Operator Methods [1], which was the starting point for many modern investigations in noncommutative analysis and quantum geometry.

1. Introduction

The problem of classical motion, as well as the problem of finding the quantum spectrum and the eigenstates of a charged particle in electric and magnetic fields of various configurations, is an inexhaustible source of topics in mathematical physics. We only note that if the notion of electric and magnetic fields is treated in an extended sense (the electric potential is understood as a Hamiltonian, and the magnetic strength tensor is understood as a symplectic structure), then this problem indeed contains any model of classical and quantum mechanics.

Such a generic nonintegrable system can be significantly simplified for several special configurations of the fields. In addition to the trivial version of the homogeneous field, the other simplest integrable cases (for three degrees of freedom) are

a. the Coulomb field of a point electric charge,

b. the Dirac field of a point magnetic charge, and

c. the Coulomb–Dirac field that is a combination of types a and b.

All these models are characterized by the same symmetry algebra so(4).

If a perturbing field is added to the central field of type a, b, or c, then the system is generally nonintegrable. Using the averaging method can asymptotically reduce this system to a system with a lower-dimensional phase space (this phase space is represented by orbits of the coadjoint representation of

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the algebra \( so(4) \), i.e., is the direct product of two spheres \( S^2 \times S^2 \). The new system is again generally nonintegrable. But this system is integrable for several special configurations of the perturbing field; precisely these cases are studied in this paper.

The fact that the averaged system has an additional integral of motion (which leads to integrability) means that instead of the algebra \( so(4) \), there is another algebra, namely, the algebra \( \mathcal{F} \) of functions on \( so(4)^* \) commuting with the additional integral of motion.

The algebra \( \mathcal{F} \) is a dynamical algebra, i.e., the Hamiltonian of the averaged system can be written via its generators (see [2], [3]). In the quantum case, the generators \( N = (N_0, \ldots, N_n) \) of the algebra \( \mathcal{F}_{\text{quant}} \) satisfy commutation relations of the form

\[
[N_j, N_k] = -i\hbar \Psi_{jk}(N). \tag{1.1}
\]

Here, the components of the (quantum) tensor \( \Psi_{jk} \) in the right-hand side are functions of generators of the algebra.

We note that it is generally impossible (except in several degenerate cases) to obtain linear functions \( \Psi_{jk} \) in relations (1.1), i.e., to reduce the problem to the Lie algebra case. Relations (1.1), as a rule, are not even algebraic. But polynomial functions \( \Psi_{jk} \) can sometimes be obtained in some coordinates. For example, the functions can be quadratic or cubic.

The configurations of the electric and magnetic fields for which relations (1.1) are polynomial are remarkable in their own way. The point is that in the polynomial case, we can construct irreducible representations for relations of form (1.1) realized by ordinary differential operators with polynomial coefficients. These operators act in the complex coordinate on symplectic leaves of the Poisson algebra corresponding to relations (1.1). As a result, the Hamiltonian of the reduced system in the cases where it is also a polynomial function of the generators \( N_j \) can be reduced (in the irreducible representation) to some second- or higher-order ordinary differential operator with polynomial coefficients. In some cases, this new operator is a model operator well-known in the theory of special functions, for example, a Heun-type operator. This operator acts in a finite-dimensional space of polynomials with respect to a complex coordinate on symplectic leaves. Its eigenfunctions are some polynomials.

We can return to the original representation by coupling these polynomials to coherent states\(^1\) and applying the quantum averaging transformation. Thus, the asymptotic approximations of the eigenfunctions of the original quantum system is given in terms of model polynomials, and the asymptotic approximations of the spectrum can be calculated explicitly using the quantum geometry [5], [6] of the symplectic leaves of algebra (1.1).

A realization of this program in connection with the motion of a particle in electromagnetic fields was first realized in the case of the Coulomb central field a perturbed by a homogeneous magnetic field. This model is called the Zeeman effect in quantum mechanics. The corresponding dynamical algebra \( \mathcal{F}_{\text{quant}} \) was obtained in [7]. Commutation relations (1.1) are quadratic in this case. They have the form

\[
[N_1, N_2] = \frac{i\hbar}{2}(N_0 N_3 + N_3 N_0), \quad [N_0, N_1] = 2i\hbar N_2,
\]
\[
[N_2, N_3] = -\frac{i\hbar}{2}(N_0 N_1 + N_1 N_0), \quad [N_0, N_2] = -2i\hbar N_1, \tag{1.2}
\]
\[
[N_3, N_1] = -\frac{i\hbar}{2}(N_0 N_2 + N_2 N_0), \quad [N_0, N_3] = 0.
\]

\(^1\)See [4] for the general properties of coherent states.