MULTIPLE FOURIER SUMS ON SETS OF $\psi$-DIFFERENTIABLE FUNCTIONS (LOW SMOOTHNESS)

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We investigate the behavior of deviations of rectangular partial Fourier sums on sets of $\psi$-differentiable functions of many variables.

1. Let $f(x) = f(x_1, \ldots, x_m)$ be a function ($f \in (T^m)$) $2\pi$-periodic in each variable and summable on the cube of periods $T^m$, let

$$S[f] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} 2^{-q(k_1,\ldots,k_m)} A_{k_1,\ldots,k_m}(f; x) = \sum_{k=0}^{\infty} 2^{-q(k)} A_k(f; x),$$

(1)

where $q(k) = q(k_1, \ldots, k_m)$ is the number of zero coordinates of the point $k = (k_1, \ldots, k_m)$ and

$$A_k(f; x) = \sum_{\gamma \in P} a_k(f; \gamma) \prod_{i=1}^{m} \cos \left( k_i x_i - \gamma_i \frac{\pi}{2} \right),$$

(2)

let $P$ be the set of all points $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m$ whose coordinates are equal to either zero or one, and let

$$a_k(f; \gamma) = \pi^{-m} \int_{T^m} f(t) \prod_{i=1}^{m} \cos \left( k_i t_i - \gamma_i \frac{\pi}{2} \right)$$

(3)

be the Fourier coefficients of the function $f(\cdot)$ corresponding to the collections $k = (k_1, \ldots, k_m)$ and $\gamma$. Combining (1)–(3), we get

$$S[f] = \sum_{k=0}^{\infty} 2^{-q(k)} \pi^{-m} \int_{T^m} f(t) \prod_{i=1}^{m} \cos k_i (t_i - x_i) dt.$$

(1')

Relations (1) and (1') are called the complete Fourier series of the function $f.$

Let $m = \{1, \ldots, m\}$, $\mu \subset m$, and let $|\mu|$ denote the number of elements of the set $\mu$. For any $f \in L(T^m)$, we set

$$S[f]_{\mu} = \sum_{k=0}^{\infty} 2^{-q(k)} \pi^{-|\mu|} \int_{T^m} f(t^\mu + x^\mu) \prod_{i \in \mu} \cos k_i (t_i - x_i) dt^\mu,$$

(4)

where \( k^\mu = (k_{j_1}, \ldots, k_{j_{|\mu|}}), \ j_1, \ldots, j_{|\mu|} \in \mu, \) and \( t^\mu = (t_1, \ldots, t_m), \ t_i = 0 \) if \( i \notin \mu, \ c^\mu = m \setminus \mu, \) and \( x^{c^\mu} = (x_1, \ldots, x_m), \ \text{and} \ x_i = 0 \) if \( i \in \mu. \)

Series (4) is called a partial Fourier series of the function \( f(\cdot) \in L(T^m) \) with respect to the group of variables \( x_i, \ i \in \mu. \) For \( \mu = m, \) we have \( S[f]|_\mu = S[f]. \)

Let \( \overline{\psi}_i(k_i) = (\psi_i^{(1)}(k_i), \psi_i^{(2)}(k_i)), \ i = 1, \ldots, m, \) be pairs of arbitrary systems of numbers \( \psi^{(j)}(k_i), \ j = 1, 2, k_i = 0, 1, \ldots, \) \( \psi_i^{(1)}(0) = 1, \ \psi_i^{(2)}(0) = 0, \ i = 1, \ldots, m. \) Assume that, for a given function \( f \in L(T^m) \) and a collection \( \mu, \) the series

\[
\sum_{k^\mu = 1}^{\infty} \pi^{-|\mu|} \int_{T^{|\mu|}} f(t^\mu + x^{c^\mu}) \prod_{i \in \mu} \psi_i^{(1)}(k_i) \cos k_i (t_i - x_i) + \psi_i^{(2)}(k_i) \sin k_i (t_i - x_i) dt^\mu, 
\]

\[
\overline{\psi}_i^{(2)}(k_i) \equiv \left(\psi_i^{(1)}(k_i)\right)^2 + \left(\psi_i^{(2)}(k_i)\right)^2 \neq 0, \ k_i \in \mathbb{N}, \ i = 1, \ldots, m,
\]

is the Fourier series of a certain function \( \varphi \in L(T^m) \) in the variables \( x_i, \ i \in \mu. \) We denote this function by \( f^\varphi(\cdot) = D^\varphi(f; x) \) and call it the \( \overline{\varphi}_\mu \)-derivative of \( f(\cdot). \) Denote the set of functions \( f \in L(T^m) \) such that, for any \( \mu \subseteq m, \) there exist the derivatives \( f^\varphi \) by \( L^\varphi = L^\varphi_{m}. \) If \( f \in L^\varphi \) and, for any \( \mu \subseteq m, \) we have \( f^\varphi \in \mathcal{R}, \) where \( \mathcal{R} \) is a certain subset of \( L(T^m), \) then we denote the set of these functions by \( L^\varphi \mathcal{R}. \)

Let \( C^\varphi \mathcal{R} \) denote the subset of continuous functions from \( L^\varphi \mathcal{R}. \)

In [1], Stepanets introduced the notion of \( \overline{\varphi}_\mu \)-derivative for \( m = 1. \) For \( m > 1, \) the definition of \( \overline{\varphi}_\mu \)-derivative for \( \mu \subseteq m \) coincides with the definition of \( (\psi, \beta)_\mu \)-derivative presented in [2, 3] in the sense that any \( \overline{\psi}_{\mu} \)-derivative coincides with a \( (\psi, \beta)_\mu \)-derivative if the parameters defining these derivatives satisfy the relations

\[
\psi_i^{(1)}(k_i) = \psi_i(k_i) \cos \beta_i \frac{\pi}{2}, \quad \psi_i^{(2)}(k_i) = \psi_i(k_i) \sin \beta_i \frac{\pi}{2}, \quad (5)
\]

and any \( (\psi, \beta)_\mu \)-derivative is also a \( \overline{\varphi}_\mu \)-derivative if equalities (5) are true.

Further, let

\[
S_n(f; x) = \sum_{k=0}^{n-1} 2^{-q(k)} A_k(f; x) = \sum_{k=0}^{n-1} 2^{-q(k)} \pi^{-m} \int_{T^m} f(t) \prod_{i=1}^{m} \cos k_i (t_i - x_i) dt, \quad n - 1 = (n_1 - 1, \ldots, n_m - 1),
\]

be a rectangular partial sum of series (1) and let

\[
\rho_n(f; x) = f(x) - S_n(f; x).
\]

In the present paper, we investigate the behavior of the deviations \( \rho_n(f; x) \) on the classes \( C^\varphi \mathcal{R}. \) For this purpose, we present several definitions (see, e.g., [1]).

Let \( \mathcal{M} \) be the set of subsequences \( \lambda_k, \ k = 0, 1, \ldots, \) convex downward and vanishing at infinity and let \( \mathcal{M}^\prime \) be the subset of \( \lambda_k \) from \( \mathcal{M} \) for which