Many statements in mathematics are given with tacit semantic assumptions about their domains of application. Consider the following statement of the Chinese Remainder Theorem which is common in many texts:

There is a solution for the congruences

\[ x \equiv a_1 \pmod{m_1} \quad \& \ldots \quad \& \quad x \equiv a_n \pmod{m_n} \]

for which \( m_1 \ldots m_n \) are relatively prime in pairs. The solution is unique for the modulus \( m_1, m_2, \ldots, m_n \).

The theorem is given with a semantic understanding that its variables range over a given intended domain (of integers). Moreover, the statement presumes we know the intended scope of the variables with respect to one another. A more proper statement of the theorem is this:

\[
(\forall m_1, \ldots, (\forall m_n)(\forall a_1, \ldots, (\forall a_n)(m_1, \ldots, m_n \ , a_1, \ldots, a_n \text{ are numbers } \& m_1 \ldots m_n \text{ are relatively prime in pairs } \rightarrow \ (\exists p)(p \equiv a_1 \pmod{m_1} \quad \& \ldots \quad \& \quad p \equiv a_n \pmod{m_n} \quad \rightarrow \quad (\forall x)(x \equiv a_1 \pmod{m_1} \quad \& \ldots \quad \& \quad x \equiv a_n \pmod{m_n} \rightarrow \ p \equiv x \pmod{m_1 \ldots m_n}))).
\]

Given we know the intended domain, some of the universal quantifiers in the formula can conveniently be dropped without undermining central scope distinctions; others cannot. We may write:

\[
(\exists p)(p \equiv a_1 \pmod{m_1} \quad \& \ldots \quad \& \quad p \equiv a_n \pmod{m_n} :\&: (\forall x)(x \equiv a_1 \pmod{m_1} \quad \& \ldots \quad \& \quad x \equiv a_n \pmod{m_n} \rightarrow \ p \equiv x \pmod{m_1 \ldots m_n})).
\]

This abandonment of the quantifiers in favor of an intended domain requires semantic assent – something altogether innocuous to mathematicians but often nocuous to philosophers.
Frege was a mathematician and a philosopher. He could not accept a mysticism of variable numbers and semantic intuitions of inferential relationships based on intended domains. But no one knew how to do a formal semantics before Tarski’s ground-breaking work in the 1930s. Semantic ideas have won the day in discussions of logic. Modern discussions begin with semantic notions against which formal deductive systems are evaluated for their completeness (or incompleteness) with respect to the wffs that remain true in every interpretation over every domain of any arbitrary cardinality. Frege was first to fully understand how a formal deductive system works so that its inferential transitions are governed solely by the syntax (shapes) of the signs. Historians do him an injustice when they imagine the deductive transitions of his formal systems to be entangled with his fledging semantic ideas. In this chapter, we shall take up some of the vexing questions of sense and reference in Frege’s Grundgesetze.

5.1 Free variables and the turnstile

Let us begin by considering the nature of derived rules in a formal system. In the Begriffsschrift, Modus Ponens (mp) and Universal Generalization are the only rules besides uniform substitution. But it is easy to formulate derived rules. For convenience of exposition, imagine the Begriffsschrift axioms in the $\text{cP} Logic of Grundgesetze (with general variables and genuine identity). Frege has:

\[\begin{align*}
\text{Axiom 1} & \quad \vDash z \\
\text{Axiom 2} & \quad \vDash y
\end{align*}\]

The rule Modus Ponens is this:

\[
\text{From } \vDash x \quad \text{and} \quad \vDash x \quad \text{infer} \quad \vDash y
\]

We can easily arrive at the derived inference rule:

\[
\text{Derived Rule from Axiom 2} \\
\text{From } \vDash x \quad \text{and} \quad \vDash x \quad \text{infer} \quad \vDash z
\]