1

Descriptive Models of Growth

1.1 Harrod-Domar model as prototype of growth models

The Harrod-Domar model of growth (Harrod 1939; Domar, 1946) is a dynamic extension of the simple real (moneyless) Keynesian model. It is built on the following assumptions:

a) production technology admits constant returns of scale. In particular, it is assumed that technology has fixed coefficients $v$ and $u$ for inputs of capital $K$ and labour $L$;

b) aggregate consumption function $C$ is the canonical long-run Keynesian consumption function. This function is assumed to have constant and identical marginal and average propensity to consume. Aggregate savings function $S$, therefore, also has constant and identical marginal and average propensity to save;

c) capital stock and labour grow over the time. Capital grows because of investments, labour grows because of population growth. Both these circumstances impart to the model a dynamic character.

Mathematically, conditions (a) ~ (c) correspond to the following equations:

\[ K = vQ \]
\[ L = uQ \]
\[ S = sQ \]
\[ \dot{K} = I \]
\[ \dot{L} = nL \]

(A)

The first and second equations of (A) define requirements of current capital $K$ and labour $L$ as proportions of current production $Q$, in
accordance with respective technical coefficients \( v \) and \( u \). The third and fourth equations define saving function \( S \), which has a constant propensity to save \( s \), and capital accumulation over time \( dK/dt = \dot{K} \) derived from current investments \( I \). The fifth equation defines changes of labour over time \( dL/dt = \dot{L} \) which is proportional to the existing labour force. The growth rate of population \( n \) is exogenously given. Equilibrium requires that both goods and labour markets clear simultaneously at every instant. This happens when investments are equal to savings \( \dot{K} = sQ \) and the labour requirement is equal to the availability of labour, \( L = \dot{L}/n = uQ \). Since \( K = vQ \), then \( v\dot{Q} = sQ \), so that equilibrium paths are described by the system of two homogeneous differential equations

\[
\dot{Q} - (s/v)Q = 0 \\
\dot{L} - nL = 0, \text{ together with the condition } L(t) = uQ(t)\psi
\]

Each equation of the system depends solely on the variable to which equilibrium refers and therefore can be solved apart. Starting from the first equation, given the initial condition \( Q(0) = Q_0 \), the equilibrium path of goods market is the exponential function \( Q(t) = Q_0 e^{(s/v)t} \). Along this path the output at every instant equals effective demand and output grows at the rate:

\[
\dot{Q}/Q = (s/v)Q_0 e^{(s/v)t}/Q_0 e^{(s/v)t} = s/v
\]

This rate is the warranted rate of growth (Harrod, 1939). An economy which grows at the warranted rate of growth has no lack of effective demand. The second equation of the differential system solves for the path of instantaneous availability of the labour force. Given an initial condition \( L(0) = L_0 \), the labour force’s availability path is the exponential function \( L(t) = L_0 e^{nt} \). The additional condition dictated by the technical requirement \( L(t) = uQ(t) \) is a restriction on the set of solutions. An output path satisfying such condition is a full-employment path. It corresponds to

\[
Q(t) = L(t)/u = (L_0/u)e^{nt} = Q_0^1 e^{nt}
\]

where \( Q_0^1 \) is the initial output which, when realized, utilizes the entire labour force \( L_0 \) at \( t_0 \). In order for an economy starting at full-employment output \( Q_0^1 \) to run full-employment forever, it must grow at the rate

\[
\dot{Q}/Q = Q_0^1 n e^{nt}/Q_0 e^{nt} = n
\]

This is the natural rate of growth (Harrod, 1939). Full employment equilibrium at every instant requires