3
Spatial Linear Regression Models

3.1 Generalities

This chapter discusses different specifications of linear spatial econometric models that can be considered once the hypothesis of no spatial autocorrelation in the disturbances is violated. A general form to take into account the violation of the ideal conditions for the applicability of OLS is given by the following set of equations:

\[ y = \lambda Wy + X\beta_{(1)} + WX\beta_{(2)} + u \quad |\lambda| < 1 \]  \hspace{1cm} (3.1)

\[ u = \rho Wu + \varepsilon \quad |\rho| < 1 \]  \hspace{1cm} (3.2)

with \( X \) a matrix of non-stochastic regressors, \( W \) a weight matrix exogenously given, \( \varepsilon \sim i.i.d.N(0, \sigma^2 I_n) \) and \( \beta_{(1)}, \beta_{(2)}, \lambda \) and \( \rho \) parameters to be estimated. The restrictions on the parameters \( \lambda \) and \( \rho \) hold if \( W \) is row-standardized.

The first equation considers the spatially lagged variable of the dependent variable \( y \) as one of the regressors and may also contain spatially lagged variables of some or all of the exogenous variables (the term \( WX \)). The second equation considers a spatial model for the stochastic disturbances. In principle, there is no need that the three weight matrices in Equations (3.1) and (3.2) are the same, although in practical cases it is difficult to justify a different choice.

Equation (3.1) can also be written as:

\[ y = \lambda Wy + Z\beta + u \quad |\lambda| < 1 \]  \hspace{1cm} (3.3)
having defined the matrix of all regressors, current and spatially lagged, as $Z = [X, WX]$ and the vector of regression parameters as $\beta = [\beta_{(1)}, \beta_{(2)}]$.

This model was termed SARAR(1,1) (acronym for Spatial AutoRegressive with additional AutoRegressive error structure) by Kelejian and Prucha (1998) and encompasses several spatial econometric models. In particular we have five remarkable cases:

(i) $\beta = 0$ and either $\lambda = 0$ or $\rho = 0$, known as the pure spatial autoregressive model
(ii) $\lambda = \rho = 0$, known as the Lagged independent variable model
(iii) $\lambda = 0$, $\rho \neq 0$ known as Spatial Lag Model (SLM)
(iv) $\lambda \neq 0$, $\rho = 0$ known as Spatial Error Model (SEM)
(v) $\lambda \neq 0$, $\rho \neq 0$ the complete model (SARAR)

We will review these five cases in the following sections. Before doing this, however, let us consider a general condition on the model’s parameters.

First of all notice that Equation (3.1) can also be written as:

\[
(I - \lambda W)y = X\beta_{(1)} + WX\beta_{(2)} + u
\]

\[
y = (I - \lambda W)^{-1}\left[X\beta_{(1)} + WX\beta_{(2)} + u\right]
\]  

(3.4)

and Equation (3.2) as:

\[
u = (I - \rho W)^{-1}\epsilon
\]  

(3.5)

provided that the two inverse matrices exist. Using the Gerschgorin (1931) theorem Kelejian and Prucha (1998) proved that, when the W matrix is row-standardized, both inverse matrices exist if $|\rho| < 1$ and $|\lambda| < 1$, hence the parameters’ restriction reported in Equations (3.1) and (3.2).

### 3.2 Pure spatial autoregression

When $\beta = 0$ and either $\lambda = 0$ or $\rho = 0$, and further assuming $\epsilon \sim i.i.d.N(0, \sigma^2 \varepsilon_n I_n)$ and W non-stochastic, then the model reduces to a simple spatial autoregression that can be estimated via the ML procedure (Whittle, 1954).