2 Living in a Black–Scholes World

Necessity is the mother of invention.

How many people in the world have heard of “the Black–Scholes model”? And how many people in the world really understand how it works? Whilst these questions are not well defined, they nonetheless illustrate a point: there is enormous variation in the depth of familiarity people have with this model. The aim of this chapter is to explore the model in sufficient depth to understand its mathematical and behavioural subtleties. Such an understanding is not only very valuable in its own right, but it also paves the way for a clearer understanding of the more advanced models which we will be covering later in the book.

2.1 The Black–Scholes model equation for spot price

If we ask ourselves what constitutes “large” or “small” moves in an FX spot price over a given time interval, we may think about moves either in terms of outright changes in spot price (“EURUSD is up a big figure”) or in terms of relative changes in spot (“EURUSD is up 1 per cent”). Which of these is more useful depends partly on the context. When focusing on a single currency pair in a single market paradigm, it is common to consider certain spot levels as special, in which case outright changes are of greater significance. However, when we wish to compare moves in the spot rates of two currency pairs, for example EURUSD and AUDJPY, it is usually more helpful to consider relative changes in spot, especially when the two spot prices have very different magnitudes.

So, should our model be based on outright or relative moves? Well, when it comes to modelling FX spot prices, we are almost always attempting to come up with a single model that describes numerous currency pairs in a given market paradigm, such as free-floating spot rates between G10 currencies. In this situation, we are not
modelling anything that depends on the importance of specific spot levels and it makes most sense to develop our model in terms of relative moves.

I wrote “almost always” in the previous paragraph, because sometimes we wish to model the idiosyncrasies of a specific currency pair, often one involving a currency which is in some way “managed,” for example using a peg. Pegged currencies, and other forms of managed currency, are discussed in Section 7.1.

The relative change in spot price is also known as the return, and that is the term that we shall use going forward.

At the heart of the Black–Scholes model for FX spot prices lies the assumption that the return follows a Brownian Motion. Named after Robert Brown [7], a botanist who in 1827 observed the random motion of pollen-produced particles when suspended in water, the term Brownian Motion is used to describe both the physical phenomenon and the mathematical equation which models it. We can describe the physical phenomenon in words by saying that the quantity in question (the position of a pollen-produced particle, or the return) wiggles about randomly, by varying amounts, and that this wiggling is possibly accompanied by an overall non-random drift at a particular speed.

Formally, we consider a time interval \((t, t + \delta t)\) and define the return as equal to the change in spot price \(\delta S\) over this time interval, divided by the initial spot price \(S\):

\[
\text{return} = \frac{\delta S}{S} \quad (2.1)
\]

Note that the return is a dimensionless quantity, since \(\delta S\) and \(S\) necessarily have the same dimensions (namely \(\text{Domestic Foreign}\)).

The assumption that the FX spot price return follows a Brownian Motion can be written as follows:

\[
\frac{\delta S}{S} = \mu \delta t + \sigma \delta W_t. \quad (2.2)
\]

We can alternatively say that the FX spot price \(S\) follows a geometric Brownian Motion. Let us now describe Equation 2.2.

The first term on the right-hand side is deterministic, with \(\mu\) a constant number, and the term represents the non-random drift part of the Brownian Motion. It is the expectation of the return over the time interval \((t, t + \delta t)\). The quantity \(\mu\) is correspondingly called either the drift rate or the expected rate of return. The term \(\mu \delta t\) needs to be dimensionless (to ensure dimensional homogeneity of Equation 2.2), and so the drift rate has the dimension of one over time, and it is measured in units of years\(^{-1}\).

In the second term on the right-hand side, the quantity \(W_t\) is called the Wiener process, and it is the standard mathematical representation of a (driftless) Brownian Motion. In words, we can say that the Wiener process represents the ‘wiggly’ part of the Brownian Motion, without any drift, and that it has its own standardized amount of wiggliness. Mathematically, the Wiener process has the