14.1 Equity

We next show how to extend the IR/FX setup to include multiple equity processes in order to cover equity derivatives. We make the simplest possible choice here and model each equity with Geometric Brownian Motion driven by the short rate-dividend yield differential, in analogy to the FX model of the previous section. The starting point is the equity process $S_k$, expressed in units of a foreign (nominal) bank account $Q^{Bi}$, where $i$ denotes the currency of equity $S_k$:

$$dS_k(t)/S_k(t) = [n_i(t) - q_k(t)]dt + \sigma^S_k(t) dW^{Bi}_{Sk}(t) \quad (14.1)$$

As a first step, our aim is to derive the dynamics of $S_k$ under the LGM numeraire $Q^{Ni}$ by finding the drift adjustment $\phi^{Ni}_{Sk}(t)$ such that

$$dW^{Ni}_{Sk}(t) = dW^{Bi}_{Sk}(t) - \phi^{Ni}_{Sk}(t) dt \quad (14.2)$$

Consider the equity asset $\hat{S}_k$ (i.e. the sum of the equity price process $S_k$ and the dividend process $q_k S_k$) expressed in units of foreign LGM numeraire. This is a tradable asset and as such is a martingale under the foreign LGM measure when expressed in units of the foreign LGM numeraire. Therefore the SDE given by

$$d \left( \frac{\hat{S}_k(t)}{N_i(t)} \right) = \frac{\hat{S}_k(t)}{N_i(t)} \left( \left[ n_i(t) + \phi^{Ni}_{Sk}(t) \sigma^S_k(t) - n_i(t) + \rho^{z_i}_{z_i} H^x_i(t) \alpha^z_i(t) \sigma^S_k(t) \right] dt 
+ \sigma^S_k(t) dW^{Ni}_{Sk}(t) - H^x_i(t) \alpha^z_i(t) dW^{N_i}_{z_i}(t) \right)$$

must have zero drift. Solving for $\phi^{Ni}_{Sk}(t)$ yields

$$\phi^{Ni}_{Sk}(t) = \rho^{z_i}_{z_i} H^x_i(t) \alpha^z_i(t) \quad (14.3)$$
Next we derive the dynamics of $S_k$ under the domestic LGM measure $Q^{N_0}$ by finding the drift adjustment $\phi_{S_k}^{N_0}(t)$ such that

$$dW_{S_k}^{N_0}(t) = dW_{S_k}^{R_k}(t) - \phi_{S_k}^{N_0}(t) dt.$$ \hspace{1cm} (14.4)$$

Firstly, inserting (14.2) and (14.3) into (14.4) yields

$$dW_{S_k}^{N_0}(t) = dW_{S_k}^{R_k}(t) - (\rho_{ii}^{zz} H_i^z(t) \alpha_i^z(t) + \phi_{S_k}^{N_0}(t)) dt.$$ \hspace{1cm} (14.5)$$

Now, the equity asset $\hat{S}_k$, converted into domestic currency and expressed in units of domestic numeraire must be a martingale under the domestic LGM numeraire. Therefore the SDE given by

$$d\left(\frac{\hat{S}_k(t) x_i(t)}{N_0(t)}\right) = \left[n_i(t) + \phi_{S_k}^{N_0}(t) \sigma_k^S(t) + \rho_{ii}^{zz} H_i^z(t) \alpha_i^z(t) \sigma_k^S(t)\right] dt$$

$$+ n_0(t) - n_i(t) + \rho_{0i}^{zz} H_0^z(t) \alpha_0^z(t) \sigma_i^S(t)$$

$$- n_0(t) + \rho_{ii}^{zz} \sigma_i^S(t) \sigma_k^S(t) - \rho_{0i}^{zz} H_0^z(t) \alpha_0^z(t) \sigma_i^S(t)$$

$$- \rho_{0i}^{zz} H_0^z(t) \alpha_0^z(t) \sigma_i^S(t) \sigma_k^S(t)$$

$$+ \sigma_k^S(t) dW_{S_k}^{N_0}(t) + \sigma_i^S(t) dW_{x_i}^{N_0}(t)$$

$$- H_0^z(t) \alpha_0^z(t) dW_{20}^{N_0}(t)$$

must be driftless. Solving for $\phi_{S_k}^{N_0}(t)$ yields

$$\phi_{S_k}^{N_0}(t) = \rho_{0i}^{zz} H_0^z(t) \alpha_0^z(t) - \rho_{ii}^{zz} H_i^z(t) \alpha_i^z(t) - \rho_{ii}^{zz} \sigma_i^S(t)$$ \hspace{1cm} (14.6)$$

and inserting (14.6) and (14.5) into (14.1) gives the dynamics of $S_k$ under the domestic LGM measure:

$$dS_k(t)/S_k(t) = \left[n_i(t) - q_k(t) + \rho_{0i}^{zz} H_0^z(t) \alpha_0^z(t) \sigma_k^S(t) - \epsilon_i \rho_{ii}^{zz} \sigma_i^S(t) \sigma_k^S(t)\right] dt$$

$$+ \sigma_k^S(t) dW_{S_k}^{N_0}(t)$$ \hspace{1cm} (14.7)$$