Survey of Integer Programming†

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The paper surveys the progress that has been made with the problem of solving linear programming problems when some or all variables are required to take integer values. It is pointed out that there are now four distinct approaches capable of solving real problems of this type: cutting plane methods, primal methods, branch and bound methods, and partial enumeration methods. Each approach is explained briefly. A detailed bibliography is attached.

1. INTRODUCTION

This paper surveys the progress that has been made with Integer Programming. This is the problem of solving linear, or possibly non-linear, programming problems when variables must take integer values. The field is divided into “pure integer programming”, when all the variables must be integers, and “mixed integer programming”, when only certain specified variables must be integers. The subject is also known as “discrete programming”, or “integer linear programming” abbreviated to “ILP”, but I think that the term integer programming is preferred by the majority of workers in the field.

A particularly important type of integer programming problem is one in which the integer variables have to be either 0 or 1, depending on whether or not some action is taken. And pure integer programming has been successfully applied to a number of combinatorial problems of this type. More generally, the problem of finding a global optimum of some function over a non-convex region can be represented as a mixed integer programming problem. This important fact was recognized by Markowitz and Manne1 in 1957. In these circumstances it does not seem important to distinguish between integer linear programming and integer non-linear programming. There is, of course, scope for studies of particular types of integer non-linear programming problems. Some pioneering work has been done in this area by Kelley,2 by Künzi and Oettli,3 and by Witzgall.4

I do not propose to say any more about applications of integer programming. The field was comprehensively surveyed by Dantzig5 in 1960, and also in his book.6

From a practical point of view, the history of the subject is, very briefly, as follows:

In 1958 Gomory devised a method, known as the Method of Integer Forms, for solving pure integer programming problems. An outline of this was published

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at the time. For a long time the main reference was a Princeton University report but this was eventually published.\(^7\) In 1960, he devised another method, the All-Integer Method.\(^8\) Several computers codes using variants of one or both methods have been written; and have successfully solved many real problems. However, their performance has not been nearly as predictable as that of ordinary linear programming codes—which are themselves rather unpredictable as regards running time. The most spectacular work in this area has been that of Glenn Martin. His code for the IBM 7094 computer uses a variant of the Method of Integer Forms called the Accelerated Euclidean Algorithm.\(^9\) It has solved a number of problems with about 100 equations and 2000 variables. Up to the beginning of 1964 the largest single problem had about 215 equations and about 2600 variables.

The field of mixed integer programming is less far advanced. Gomory has extended his method of integer forms to deal with continuous as well as integer variables.\(^10\) Various computer codes have been written using this method. Reports on their performance have varied from good to disappointing, but I am not aware that they have solved any large problems. William White has shown that, unless the objective function itself must be integral, the method may not terminate.

A mixed integer programming procedure, published by Healy\(^11\) under the title Multiple Choice Programming has been programmed for the IBM 7090 computer and found to solve practical problems, although its theoretical status is obscure. Its emphasis on problems in which a sum of non-negative integer-valued variables must add up to 1 is appropriate, since many practical problems have this structure.

Driebeck\(^12\) and Dakin\(^13\) have developed programs using a "branch and bound" method for mixed integer programming.

2. ALGORITHMS

The rest of this paper will be devoted to algorithms for both pure and mixed integer programming. I have classified these algorithms according to the type of approach taken. Since many people, including me, thought until 1958 that a general method of solving integer programming problems was self-evidently impossible, it is remarkable that there are now four distinct approaches capable of solving real problems in this area effectively. These can be described briefly as:

- Cutting plane methods
- Primal methods
- Branch and bound methods
- Partial enumeration methods.

Since pure integer programming seems to be easier than mixed integer programming, methods of reducing a mixed problem to a pure one are