FURTHER COMMENT ON: A CRITERION FOR DECISION-MAKING

With reference to a paper by G. Hayhurst\(^1\) and a comment on it by D. J. White\(^2\), I would like to make a further comment on their use of the maximin criterion to the matrix \(U\) which was derived from the original payoff matrix \(A\) by exploiting a partial information one player has about the opponent's strategy.

Let \(A = [a_{ij}]\) be the \(m \times n\) payoff matrix of a two-person zero-sum game. Suppose the situation for player I is one in which he has sufficient reason to assume that his opponent, player II, will adopt a specified mixed strategy \(y^0\) with probability \(\beta\), and some other completely unpredictable strategy \(y\) with probability \(1 - \beta\). This assumption will be sound, for instance, if:

(a) II is unable to find his optimal strategy for the matrix game \(A\), because of his unfamiliarity with game theory, so that he persists to use an easy-to-see strategy such as \(<1/n, \ldots, 1/n>\) or some other non-optimal substitute for the optimum.

or

(b) II has some political reason of his persistence to a particular strategy \(y^0\), which may be different from the optimal strategy, and I is aware of this fact.

Under the above assumption, then, the problem for I becomes one of maximizing (in \(x\))

\[
\min_{y} xA[\beta y^0 + (1 - \beta)y],
\]

where mixed strategies \(x\) for I, and \(y\) and \(y^0\) for II, are written by row and column vectors, respectively. We define

\[
U = [u_{ij}], \quad u_{ij} = \beta(Ay^0)_i + (1 - \beta)a_{ij},
\]

where the subscript \(i\) denotes the \(i\)th component of the vector. \(G_i\) defined by Hayhurst\(^1\) is

\[
\min_j u_{ij} = \beta(Ay^0)_i + (1 - \beta) \min_j a_{ij}.
\]

Since \(xA[\beta y^0 + (1 - \beta)y] = xUy\), player I is asked to find a max-min strategy for \(U\). Professor White commented that (a) in general, a max-min strategy for \(U\) is not max-min for \(A\), nor is pure, and (b) a pure strategy \(i\) is preferred to the max-min strategy \(x^*\) for \(A\), iff \(G_i > \beta x^* Ay^0 + (1 - \beta)v(A)\), where \(v(A)\) is the value of the game \(A\).

The inequalities below the value of the game \(U\) deserve consideration. Let us write \(U\) as \(U_{y^0, \beta}\) to emphasize that it depends on \(y^0\) and \(\beta\). Then we have

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\]
Viewpoints

Theorem 1

\[ v(A) \leq v(U_{y^0, p}) \leq \beta \max_i (A y^0)_i + (1 - \beta) v(A), \]  

for any strategy \( y^0 \) by II and any \( 0 \leq \beta \leq 1 \). Hence if \( y^0 \) coincides with a min-max strategy \( y^* \) for \( A \), then for any \( 0 \leq \beta \leq 1 \) we have

\[ v(U_{y^*, p}) = v(A). \]  

Proof:

Since \( v(U) = \max_x \min_j (x U)_j = \max_x \min_j \sum_i x_i \{ \beta (A y^0)_i + (1 - \beta) a_{ij} \} \)

\[ = \max_x \{ \beta x A y^0 + (1 - \beta) \min_j (x A)_j \} \]

\[ \geq \beta x^* A y^0 + (1 - \beta) v(A) \]

\[ \geq \beta v(A) + (1 - \beta) v(A) = v(A), \]

and

\[ v(U) = \min_y \max_i (U y)_i = \min_y \max_i \{ \beta (A y^0)_i + (1 - \beta) (A y^0)_i \} \]

\[ \leq \min_y \{ \beta \max_i (A y^0)_i + (1 - \beta) \max_i (A y)_i \} \]

\[ = \beta \max_i (A y^0)_i + (1 - \beta) \min_i \max_y (A y)_i \]

\[ = \beta \max_i (A y^0)_i + (1 - \beta) v(A), \]

we obtain (1). Moreover since

\[ \max_i (A y^*)_i = v(A) \]

if \( y^* \) is min-max for \( A \), we obtain (2) from (1).

We can interpret the left half of the inequalities in (1) as expressing the fact that player I will be more profitable than in \( A \) by exploiting his partial information (i.e., \( y^0 \) and \( \beta \)) about his opponent’s strategy. If \( y^0 \) is different from \( y^* \) and \( \beta \) is near unity, the max-min strategy for \( U_{y^0, p} \) is very likely to be pure.

Theorem 2

If \( y^0 \) coincides with \( y^* \) with probability \( \beta \), however small, then any max-min strategy \( x^* \) for \( A \) is also max-min for \( U_{y^*, p} \).