A Note on the Joint Replenishment Problem under Constant Demand

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This note considers the joint replenishment inventory problem for \( N \) items under constant demand. The frequently-used cyclic strategy \( (T; k_1, \ldots, k_N) \) is investigated: a family replenishment is made every \( T \) time units and item \( i \) is included in each \( k_i \)th replenishment. Goyal proposed a solution to find the global optimum within the class of cyclic strategies. However, we will show that the algorithm of Goyal does not always lead to the optimal cyclic strategy. A simple correction is suggested.

Key words: inventory, joint replenishment

INTRODUCTION

Joint replenishment of a family of items is an extensively studied topic in inventory theory. The problem is to construct a replenishment strategy for (a family of) \( N \) items which interact because of a special set-up cost structure: a major set-up cost is incurred for each joint replenishment, independent of which items are involved. In addition, a minor set-up cost is incurred for each item included in the replenishment. So, cost savings can be achieved by coordinating the replenishments of several items. Refer to Aksoy and Erenguc\(^1\) and Goyal and Satir\(^2\) for a more detailed introduction to the problem.

In this paper we consider the joint replenishment problem under constant demand. A commonly-used strategy is to place an order every \( T \) time units and to include the \( i \)th item into every \( k_i \)th replenishment. This leads to a replenishment interval of \( T \cdot k_i \) time units for the \( i \)th item. Note that (actual) joint replenishments are equally spaced under such a strategy only if \( k_{\min} = 1 \), where \( k_{\min} \) is the minimum value of \( k_i \) over all items \( i \). The cyclic strategies \( (T; k_1, \ldots, k_N) \) with \( k_{\min} = 1 \) are called strict cyclic.

Example 1

Let \( N = 2, k_1 = 2, k_2 = 3, T = 1 \). Table 1 shows the items which are included in the order on subsequent time intervals. Note that no order is placed on \( t = 1 \) and \( t = 5 \). The actual joint replenishments are not equally spaced.

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>[1,2]</td>
<td>[ ]</td>
<td>[1]</td>
<td>[2]</td>
<td>[1]</td>
<td>[ ]</td>
<td>[1,2]</td>
</tr>
</tbody>
</table>

Andres and Emmons\(^3\) have shown by a counter example that the overall optimal strategy for the joint replenishment problem is not necessarily a cyclic strategy. They illustrate this statement with a two-product example, for which the optimal solution is obtained by a special algorithm.\(^4\)

Moreover, the problem setting of Andres and Emmons is different from that of the joint replenishment problem in the following sense: a major set-up cost has to be paid at every replenishment in which not all items are involved. The only way to avoid the major set-up cost is to perform a joint replenishment for all items simultaneously (the problem settings are only equivalent if the number of items is two).

Among others, Chakravarty and Goyal\(^5\) considered a class of strategies under which the items

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of the family are divided into a number of groups. The items of each group have the same replenishment cycle, but the replenishment cycles of the groups are not an integer multiple of the shortest (basic) cycle. These strategies are referred to as grouping strategies. Chakravarty and Goyal conclude that the class of cyclic strategies is inferior to the class of grouping strategies if the coefficient of variation of the EOQ replenishment frequencies of the items is low. Van Eijjs et al. compared the performance of cyclic and grouping strategies with the help of an extensive simulation experiment. It turned out that the class of non-cyclic grouping strategies outperforms the cyclic \((T; k_1, \ldots, k_N)\) strategies in situations in which the ratio of the major set-up cost and the average minor set-up cost is low (lower than 0.5, depending on the number of items in the family).

Another class of strategies which permit unequal time spacing between joint replenishments is suggested by Goyal and Soni. They provide an extension of the \((T; k_1, \ldots, k_N)\) strategies by permitting multiple basic cycles.

In the following, the class of cyclic strategies is considered. Goyal has proposed an algorithm to find the optimal solution within this class of strategies. We will show that the solution, which is found by this algorithm, is optimal within the class of strict cyclic strategies, but is not necessarily optimal within the class of cyclic strategies. We propose to adjust the algorithm of Goyal slightly to make it possible to obtain cyclic strategies with \(k_{\min} > 1\).

**THE JOINT REPLENISHMENT PROBLEM**

The joint replenishment problem can be described as follows: \(N\) items are purchased from the same supplier. When one or more of the items of the family are ordered a major set-up cost \((A)\) is incurred, regardless of which items are ordered. In addition, a minor set-up cost \((a_i)\) is charged for each particular item which is included in the replenishment. Item \(i\) has constant demand \((D_i)\) per time unit and holding cost \((h_i)\) per dollar per time unit. Stock-outs are forbidden and the rate of replenishment is assumed to be infinite. The objective is to minimize the total cost per time unit over an infinite horizon. For an arbitrary cyclic strategy \((T; k_1, \ldots, k_N)\) the total average cost per time unit is given by

\[
TRC(T; k_1, \ldots, k_N) = \frac{1}{T} \left( A + \sum_{i=1}^{N} \frac{a_i}{k_i} \right) + \frac{T}{2} \sum_{i=1}^{N} D_i h_i k_i.
\]  

(1)

We assume that for a cyclic strategy with \(k_{\min} > 1\) the major set-up cost is also incurred at multiples of \(T\) at which no actual replenishment is performed. Dagpunar reformulated the objective function of the \((T; k_1, \ldots, k_N)\) strategies under the assumption that no major set-up cost is charged if no order is placed. However, as pointed out by Goyal, the minimization of the cost function, provided by Dagpunar, is considerably more complex than that of the original problem.

For fixed \((k_1, \ldots, k_N)\) the optimal value \(T^*(k_1, \ldots, k_N)\) of the basic cycle equals:

\[
T^*(k_1, \ldots, k_N) = \left[ \frac{2 \left( A + \sum_{i=1}^{N} \frac{a_i}{k_i} \right)}{\sum_{i=1}^{N} D_i h_i k_i} \right]^{1/2}.
\]

(2)

If \(T^*(k_1, \ldots, k_N)\) is substituted back in the original cost function (1) the following cost function is obtained:

\[
TRC(k_1, \ldots, k_N) = \left[ \frac{2 \left( A + \sum_{i=1}^{N} \frac{a_i}{k_i} \right) \left( \sum_{i=1}^{N} D_i h_i k_i \right)}{\sum_{i=1}^{N} D_i h_i k_i} \right]^{1/2}.
\]

(3)

Most solution procedures solve a subproblem in which the \(k_i\)s are not necessarily restricted to integer variables. Schweitzer and Silver have shown for this continuous variable case that the problem is ill-posed if the restriction \(k_i \geq 1\) is deleted. It can easily be shown (see Van Eijjs) that under this restriction, \(k_{\min} = 1\) in the optimal solution of the continuous variable case. However,