Magnetization of Type-II Superconductors in the Range of Fields $H_{c1} \leq H \leq H_{c2}$: Variational Method

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Abstract—A variational method is proposed to find the magnetic field dependence of the magnetization of type-II superconductors in the mixed state by a self-consistent technique. This model allows for suppression of the order parameter to zero at the centers of Abrikosov vortices and also for the magnetic field dependence of the order parameter. The results can be applied to the entire range of fields $H_{c1} \leq H \leq H_{c2}$ for any values of the Ginzburg–Landau parameter $\kappa > 1/\sqrt{2}$. It is shown that in weak fields where $\kappa \gg 1$ the behavior of the magnetization can be described exactly in the London approximation provided that the correct value of $H_{c1}$ is used. Near the second critical field this dependence shows good agreement with the well-known Abrikosov result. It is also shown that using the concept of isolated vortices and applying the principle of superposition of the fields and currents generated by these vortices to calculate the magnetization gives inaccurate quantitative results even in fairly weak fields. By going beyond these concepts, it was possible to allow more accurately for the influence of the vortex cores on the magnetization behavior in the intermediate range of fields $H_{c1} \leq H \leq H_{c2}$ and to identify the range of validity of various approximations used widely in the literature. © 2000 MAIK “Nauka/Interperiodica”.

1. INTRODUCTION

The magnetization of type-II superconductors is a fundamental electromagnetic characteristic. It can be used to find various important parameters of the superconductor such as the lower $H_{c1}$ and upper $H_{c2}$ critical fields, and the Ginzburg–Landau parameter $\kappa$ [1–3]. An enormous number of experimental and theoretical studies have been devoted to magnetization (see, for example, the reviews [4, 5]). In this context it is important to obtain formulas for the magnetization of superconductors which would be suitable for quantitative calculations over a wide range of external magnetic fields. This problem has been discussed in the literature for some time (see, for example, [2–15]). Until recently, however, there was no convenient and reliable approach which could be applied to calculate the magnetization of a type-II superconductor analytically over the entire range of external fields $H_{c1} \leq H \leq H_{c2}$.

The problem of calculating the magnetic moment $M$ of a superconductor can be solved most easily in weak fields $H \ll H_{c2}$. Here the cores of the Abrikosov vortices occupy only a small part of the volume and $M(H)$ is obtained for $\kappa \gg 1$ using the London approximation where the modulus of the order parameter is assumed to be constant to calculate the local fields and currents outside the core [1–3]. In the London model the dependence of the magnetization $M$ of an ideal isotropic superconductor on the magnetic field $H$ in the range $H_{c1} \leq H \leq H_{c2}$ can be described using the Fetter formula [6]:

$$-4\pi M = H_{c1} - \frac{1}{4\kappa}\left\{ \ln\left[2\kappa(H - H_{c1})\right] + 1.34 \right\}. \tag{1}$$

In this formula and subsequently we use a system of units [3] in which all the distances are normalized to the London depth of penetration of the magnetic field $\lambda$, the magnetic field is normalized to $H_{c1}/\sqrt{2}$ (where $H_{c1}$ is the thermodynamic critical field), the order parameter is normalized to its equilibrium value, and the vector potential is normalized to $\Phi_0 = 2\pi\kappa$ and $H_{c2} = \kappa$. The lower critical field $H_{c1}$ cannot be calculated self-consistently in the London model. For this reason, $H_{c1}$ appears in Eq. (1) as a parameter and for $\kappa \gg 1$ may be written in the form [3]

$$H_{c1} = \frac{1}{2\kappa}(\ln \kappa + \varepsilon). \tag{2}$$

The constant $\varepsilon$ is determined by the structure of the order parameter at the vortex core and its value $\varepsilon \approx 0.50$ was determined by Hu [7] by means of a numerical solution of the complete Ginzburg–Landau system of
equations (see also [8]). The Fetter dependence (1) differs for \( H \rightarrow H_{c1} \). In the immediate vicinity of \( H_{c1} \) the magnetization in the London model can be obtained numerically [14] or analytically using an approximation which only allows for vortex interaction with nearest neighbors in the vortex lattice [3]. In order to extend the validity of the London approximation, various approaches have been developed which make partial allowance for the contribution of the vortex cores to the free energy of the superconductor (see [5, 13]).

The London model cannot be applied in strong fields because the vortex density is high in this case. The behavior of the magnetization near the second critical field is described by the well-known Abrikosov expression [3]:

\[
M = \frac{H - H_{c2}}{4\pi \beta_\alpha (2\kappa^2 - 1)}, \quad H_{c2} - H \ll H_{c2}, \tag{3}
\]

where, for a triangular vortex lattice, we have \( \beta_\alpha = 1.16 \).

In [10, 11] Clem proposed a fairly simple variational model which allows for the structure of the order parameter near the center of the vortex. The following trial function was used for the modulus of the order parameter:

\[
f = \frac{f_\alpha r}{\sqrt{r^2 + \xi^2}}, \tag{4}
\]

where \( r \) is the distance from the center of the vortex, \( \xi \) and \( f_\alpha \) are variational parameters characterizing the spatial distribution of the order parameter. This model was used to obtain a formula for \( H_{c1} \) [10,11] which for \( \kappa \gg 1 \) may be expressed in the form (2) where \( \varepsilon = 0.52 \) which shows good agreement with the results of [7, 8].

Hao and Clem then generalized this variational model to the case of a regular vortex lattice and obtained a unified formula for \( M(H) \) which can be applied over the entire range of fields \( H_{c1} \leq H \leq H_{c2} \) [11]. One of the most important conclusions of this study is that, even in weak fields, the influence of the vortex cores cannot be neglected and consequently the London model cannot generally give an exact result [11,12]. In the range of fields near \( H_{c2} \) the dependence \( M(H) \) obtained in [11] is almost the same as the Abrikosov result (3). This theory was subsequently generalized to the case of anisotropic superconductors [16]. The model proposed in [11] has been widely used in the literature. The formula for the magnetization has been actively used to analyze experimental data from measurements of the magnetic moment of various superconductors such as: YBa\(_2\)Cu\(_3\)O\(_{7-\delta}\) [17], YBa\(_2\)Cu\(_3\)O\(_5\) [18], Bi\(_2\)Sr\(_2\)Ca\(_2\)Cu\(_3\)O\(_{10}\) [19], (Tl,Pb)Sr\(_2\)Ca\(_2\)Cu\(_3\)O\(_7\) and Tl\(_2\)Ba\(_2\)Ca\(_2\)Cu\(_3\)O\(_{10}\) [20], HgBa\(_2\)Ca\(_2\)Cu\(_3\)O\(_{7+\delta}\) [21, 22], Hg\(_{0.98}\)Pb\(_{0.02}\)Ba\(_{1.5}\)Sr\(_2\)Cu\(_3\)O\(_{8-\delta}\) [23], and Nd\(_{1.85}\)Ce\(_{0.15}\)CuO\(_{4-\delta}\) [24].

In the present paper we show that several errors were made in the derivation of the formula for \( M(H) \) in [11]. For example, the expression for the free energy of the vortex lattice \( F \) was formulated using the principle of superposition of the fields and currents of isolated vortices, which can only be applied in weak fields. At the same time, also in the calculations of \( F \) the transition was made from summation over the reciprocal vortex lattice to integration. The error associated with this transition and also using an inaccurate value of \( H_{c1} \) in (1) led these authors [11] to the erroneous conclusion that the London model is incorrectly formulated for \( \kappa \gg 1 \) even in weak fields. Unfortunately, this statement is now accepted by a whole range of researchers. Additionally, the dependences of the variational parameters \( \xi_\nu \) and \( f_\alpha \) on the magnetic induction given in [11] (which determine the behavior of the magnetization to a considerable extent) were not obtained self-consistently and do not follow from the expression used for the free energy, but are simply convenient approximations.

Here we use a variational model to obtain a self-consistent derivation of the expression for \( M(H) \). In this case, the spatial distribution of the order parameter was simulated using the trial function (4) and the unit cell of the regular vortex lattice was replaced by a circular one (Wigner–Seitz approximation). The formula obtained for \( M(H) \) can be applied over the entire range of fields \( H_{c1} \leq H \leq H_{c2} \) for any values of the Ginzburg–Landau parameter \( \kappa > 1/\sqrt{2} \). The result for the magnetization in weak fields for \( \kappa \gg 1 \) agrees with the London dependence (allowing for the exact value of \( H_{c1} \)) whereas in strong fields it shows good agreement with the Abrikosov result. The formula obtained for the magnetization can easily be generalized to the case of anisotropic superconductors where the vortices are oriented along one of the principal axes of the crystal. For this orientation a scaling transformation exists which can be used to calculate the magnetization of an anisotropic superconductor from an isotropic one simply by changing the notation of \( \kappa \) [11]. This aspect is considered in Section 2.

We also discuss the correctness of the approximation of isolated vortices in the mixed state of a superconductor. It is shown that even in weak fields, when the density of vortex filaments is still low, using the principle of superposition of the fields created by separate vortices leads to appreciable quantitative errors in calculations of the magnetization.

2. WIGNER–SEITZ APPROXIMATION

In weak fields the distances between the neighboring vortex filaments are many times the dimensions of the vortex cores. This means that the vortices can be considered as independent interacting objects (see, for example, [1–3, 13, 25, 26]). Thus, in weak fields the principle of field superposition is satisfied: the self-induced field of each filament is assumed to be the same as that of an isolated filament and the local field at an