Stochastic Transport in Random Wave Fields

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Abstract—An analysis is made of particle diffusion and the field of a passive impurity in random wave fields. A characteristic of this problem is that the statistical transport coefficients (diffusion coefficients) vanish in the approximations normally used (delta-correlated random field or diffusion) giving the Fokker–Planck equation. In this study perturbation theory is used in the first nonvanishing order of smallness which allows these transport coefficients to be calculated for waves of various types. © 2000 MAIK “Nauka/Interperiodica”.

1. INTRODUCTION. CHARACTERISTICS OF PARTICLE DIFFUSION IN RANDOM WAVE FIELDS OF VELOCITY AND EXTERNAL FORCES

Particle motion in rapidly varying random velocity fields or under the action of rapidly varying random forces is an important problem having numerous applications in mechanics, hydrodynamics, plasma physics, and so on. It is well known that stochastic transport in rapidly-varying vibrational and wave fields leads to various important physical phenomena such as Fermi acceleration, stochastic plasma heating, and so on [1, 2]. These phenomena are generally described using the Fokker–Planck equation whose coefficients are expressed in terms of the correlation functions of random fields and are calculated using methods of averages developed for nonlinear equations. Although the results thus obtained reflect the main features of these phenomena, they do not have a universal character or a clearly defined range of validity.

At the same time we know that a broad class of problems can be described fairly comprehensively using the approximation of a delta-correlated random process, the diffusion approximation, or various generalizations of these based on a functional technique with variational derivatives (see, e.g., [3–7]). Calculations of the transport coefficients for random wave fields indicate that in many cases these may have values of the second order of smallness relative to the values appearing in ordinary variants of the theory of short-correlated random fields. It is therefore of considerable interest to develop a general method of deriving equations for the statistical characteristics of stochastic particle transport and fields and to calculate the transport coefficients taking into account terms of the second order of smallness if the coefficients of the first order of smallness vanish.

A general method for such calculations was proposed in [8] (see, also [3, 9]). This method can be used to analyze a range of phenomena described by different authors using different approaches from a common viewpoint, it can be used to calculate the statistical characteristics of particle ensembles and fields, and also to indicate the ranges of validity of the equations obtained.

Particle diffusion in the random velocity field $\mathbf{u}(r, t)$ is usually described using the first-order differential equation

$$
\frac{d}{dt} \mathbf{r}(t) = \mathbf{u}(r, t), \quad \mathbf{r}(0) = \mathbf{r}_0.
$$

(1)

Particle diffusion in a field of random external forces $\mathbf{f}(r, t)$ with linear friction is described by the system of equations

$$
\frac{d}{dt} \mathbf{r}(t) = \mathbf{v}(t), \quad \frac{d}{dt} \mathbf{v}(t) = -\lambda \mathbf{v}(t) + \mathbf{f}(r, t),
$$

(2)

$$
\mathbf{r}(0) = \mathbf{r}_0, \quad \mathbf{v}(0) = \mathbf{v}_0.
$$

We introduce some indicator functions for Eqs. (1) and (2)

$$
\varphi(r, t) = \delta(\mathbf{r}(t) - \mathbf{r}),
$$

(3)

$$
\varphi(r, v, t) = \delta(\mathbf{r}(t) - \mathbf{r})\delta(\mathbf{v}(t) - \mathbf{v}),
$$

which are described by the Liouville equations (see, e.g., [3])

$$
\frac{\partial}{\partial t} \varphi(r, t) = \frac{\partial}{\partial \mathbf{r}} \{ \mathbf{u}(r, t) \varphi(r, t) \},
$$

(4)

$$
\varphi(r, 0) = \delta(\mathbf{r} - \mathbf{r}_0),
$$

$$
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{v}} - \lambda \frac{\partial}{\partial \mathbf{v}} \right) \varphi(r, \mathbf{v}, t)
$$

$$
= -\mathbf{f}(r, t) \frac{\partial}{\partial \mathbf{v}} \varphi(r, \mathbf{v}, t)
$$

$$
\varphi(r, \mathbf{v}, 0) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(\mathbf{v} - \mathbf{v}_0).
$$
The average of the indicator function \( \phi(r, t) \) over the ensemble of realizations of the random field \( \{u(r, t)\} \) will then describe the single-point probability density of the particle position,

\[
P(r, t) = \langle \phi(r, t) \rangle_u = \langle \delta(r(t) - r) \rangle_u.
\]

and the average of the indicator function \( \phi(r, v, t) \) over the ensemble of realization of the random field \( \{f(r, t)\} \) will describe the joint single-point probability density of the particle position and its velocity

\[
P(r, v, t) = \langle \phi(r, v, t) \rangle_f = \langle \delta(r(t) - r)\delta(v(t) - v) \rangle_f.
\]

We shall average equation (4) over the ensemble of realizations of the random fields \( \{u(r, t)\} \) and \( \{f(r, t)\} \). As a result we obtain the open equations

\[
\frac{\partial}{\partial t} P(r, t) = \frac{\partial}{\partial r} \langle u(r, t) \phi(r, t) \rangle,
\]

\[
P(r, 0) = \delta(r - r_0),
\]

\[
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} - \lambda \frac{\partial}{\partial v} \right) P(r, v, t)
\]

\[
= - \delta \langle f(r, t) \phi(r, v, t) \rangle,
\]

\[
P(r, v, 0) = \delta(r - r_0) \delta(v - v_0),
\]

containing the correlations \( \langle u(r, t) \phi(r, t) \rangle \) and \( \langle f(r, t) \phi(r, v, t) \rangle \). We shall assume that the fields \( u(r, t) \) and \( f(r, t) \) are Gaussian random fields, spatially uniform and steady-state in time having zero averages and correlation tensors

\[
B^{(u)}_{ij}(r - r', t - t') = \langle u_i(r, t) u_j(r', t') \rangle,
\]

\[
B^{(f)}_{ij}(r - r', t - t') = \langle f_i(r, t) f_j(r', t') \rangle.
\]

The correlations can then be split using the Furutsu–Novikov formula (see, e.g., [3, 6, 10])

\[
\langle f_i(x, t) R[t; f(y, \tau)] \rangle
\]

\[
= \int dx' \int dt' \delta_{ij} B_{jk}(x, t; x', t') \left\langle \frac{\delta}{\delta f_k(x', t')} R[t; f(y, \tau)] \right\rangle,
\]

which holds for the Gaussian random field \( f(x, t) \) with the arbitrary functional \( R[t; f(y, \tau)] \) of it. Consequently, Eqs. (5) may be rewritten in the form

\[
\frac{\partial}{\partial t} P(r, t) = - \frac{\partial}{\partial r} \int_0^t dt' \frac{\partial}{\partial r} P^{(u)}_{ij}(r - r', t - t')
\]

\[
\times \left\langle \frac{\delta}{\delta u_i(r', t')} \phi(r, t) \right\rangle_u,
\]

\[
P(r, 0) = \delta(r - r_0),
\]

\[
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} - \lambda \frac{\partial}{\partial v} \right) P(r, v, t)
\]

\[
= \frac{\partial}{\partial v} \left[ \frac{\partial}{\partial r} \phi(r, v, t) \right] P(r, v, t) + \frac{\partial}{\partial v} \phi(r, v, t) P(r, v, t),
\]

\[
P(r, v, 0) = \delta(r - r_0) \delta(v - v_0).
\]

We subsequently need to use asymptotic methods to obtain closed equations. The simplest of these methods are the approximation that the random fields \( u(r, t) \) and \( f(r, t) \) are delta-correlated in time, and the diffusion approximation.

1.1. Delta-Correlated Approximation

In the approximation that the random fields \( u(r, t) \) and \( f(r, t) \) are delta correlated in time, the correlation tensors (6) are approximated by the expressions

\[
B^{(u)}_{ij}(r - r', t - t') = 2 B^{(u)}_{ij} \delta(t - t'),
\]

where

\[
B^{(u)}_{ij} = \frac{1}{2} \int dt B^{(u)}_{ij}(r, t) = \int dt B^{(u)}_{ij}(r, t).
\]

Now taking into account the equalities

\[
\frac{\delta \phi(r, t)}{\delta u_i(r', t')} \bigg|_{t = t'} = - \frac{\partial}{\partial r_i} \{ \delta(r - r') \phi(r, t') \},
\]

\[
\frac{\delta r_i(t)}{\delta f_j(r', t')} \bigg|_{t = t'} = 0,
\]

\[
\frac{\delta v_i(t)}{\delta f_j(r', t')} \bigg|_{t = t'} = \delta_{ij} \delta(r(t') - r'),
\]

derived from (4) and (2), Eqs. (8) may be rewritten in a closed form corresponding to the Fokker–Planck equation

\[
\frac{\partial}{\partial t} P(r, t) = D^{(u)}_{ij} \frac{\partial^2}{\partial r_i \partial r_j} P(r, t),
\]

\[
P(r, 0) = \delta(r - r_0),
\]

\[
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} - \lambda \frac{\partial}{\partial v} \right) P(r, v, t)
\]

\[
= D^{(f)}_{ij} \frac{\partial^2}{\partial v_i \partial v_j} P(r, v, t),
\]

\[
P(r, v, 0) = \delta(r - r_0) \delta(v - v_0).
\]