New Recoil-Effect-Induced Contributions to the Fine Shift of $S$ Levels in the Muonium Atom

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Abstract—A new $\alpha^6 \mu^3 / (m_1 m_2) \ln (m_1 m_2)$ correction that arises in one-photon particle interaction in the muonium atom is found and calculated. © 2001 MAIK “Nauka/Interperiodica”.

Although relativistic two-body theory is of interest in and of itself, its application to calculating the structure of the energy levels in hydrogen-like atoms is also of paramount importance. Since such atoms are the most accessible to both theoretical and experimental investigations, they offer the possibility of testing the fundamentals of quantum theory. In addition, a comparison of theoretical and experimental spectroscopic data on hydrogen-like atoms can be used to refine the value of the fine-structure constant $\alpha$, which is expressed in terms of the universal fundamental constants. Finally, the results of such investigations can play an important role in practical applications like metrology.

In a number of recent studies, logarithmic corrections to the fine shift of the $S$ levels in hydrogen-like atoms were considered to the sixth order in the constant $\alpha$. For the sake of convenience, we represent these corrections in the form

$$
\Delta E_{nS} (\alpha^6) = \frac{\alpha^6 \mu^3}{m_1 m_2} \left[ C_0 \ln \beta^{-1} + C_1 \ln \alpha^{-1} + C_2 \beta \ln^2 \beta \right],
$$

where $\beta = \frac{m_1}{m_2}$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

(1)

It was shown in [1, 2] that

$$
C_1 = \begin{cases} 
0 & \text{for } m_1 \neq m_2 \\
\frac{1}{32} \alpha^6 \mu \delta_{10} n^{-3} & \text{for } m_1 = m_2 = m.
\end{cases}
$$

In [3], we considered the coefficient $C_2$; here, we are going to calculate $C_0$.

Our consideration is based on the quasipotential equation [4, 5]

$$
(E - \varepsilon_{1p} - \varepsilon_{2p}) \Psi_E (p, E) = (2\pi)^{-3} \int d^3q V_E (p, q, E) \Psi_E (q, E),
$$

where $\varepsilon_{1p} = \sqrt{p^2 + m_1^2}$ and $V_E$ is the quasipotential given by

$$
V_E = F_E^{-1} - \left(\hat{G}_E^+\right)^{-1}.
$$

Here, we have

$$
F_E^{-1} = (2\pi)^3 \delta^3 (p - q) \left( E - \varepsilon_{1p} - \varepsilon_{2p} \right),
$$

$$
\left[ \ldots \right]^+ = u^+_i \bar{u}^+_j \left[ \ldots \right] \gamma_{10} \gamma_{20} u_i u_j,
$$

$u_i$ is the Dirac spinor of the $i$th particle, and

$$
\hat{G} (p, q, E) = (2\pi)^{-2} \int G (p_0, q_0, p, q, E) \, dp_0 dq_0.
$$

(3)

We now isolate the interaction with a Coulomb kernel in the original equation and denote by $\Delta E_n$ the correction to the energy level $E_n$ of the nonrelativistic Schrödinger equation with a Coulomb potential:

$$
\left( \frac{F_E^{-1} + \Delta E_n - K^+ - \tilde{V}_E}{\Delta E_n} \right) \Psi_E = 0,
$$

$$
\Delta E_n = E - E_n, \quad \tilde{V}_E = V_E - K^+,
$$

$$
K^+ (p, q) = \epsilon_1 \epsilon_2 \gamma_{10} \gamma_{20} (p - q)^{-2}.
$$

In the nonrelativistic equation, the energy eigenvalues for the Coulomb interaction and the corresponding eigenfunctions are determined by the expressions

$$
E_n = m_1 + m_2 - \frac{\mu \alpha^2}{2m^2},
$$

$$
E_n = E_{1n} + E_{2n} = \eta_1 E_n + \eta_2 E_n,
$$

$$
\eta_1 = \frac{E_{1n}^2 + m_1^2 - m_2^2}{2E_n^2}, \quad \eta_2 = \frac{E_{1n}^2 + m_2^2 - m_1^2}{2E_n^2},
$$

$$
E_{1n} = m_1 - \frac{\mu^2 \alpha^2}{2m_1 n^2}.
$$

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\[ E_{2n} = m_2 - \frac{\mu^2 \alpha^2}{2m_2 n^2}, \quad m_1^2 - E_{1n}^2 \]  
\[ = m_2 - E_{2n} = \frac{\mu^2 \alpha^2}{n^2}, \]  
\[ \Psi_{1S}(p) = \varphi_1(0) \frac{8\pi \omega}{(p^2 + \omega^2)^2} |\varphi_1(0)|^2 = \frac{\omega^3}{\pi}, \]  
\[ \omega = \mu \alpha, \]  
\[ \Psi_{2S}(p) = \varphi_2(0) \frac{16\pi \Omega}{(p^2 + \Omega^2)^2} |\varphi_2(0)|^2 = \frac{\Omega^3}{\pi}, \]  
\[ \Omega = \frac{1}{2} \omega. \]  

Since
\[ (p^2 + \omega^2) = (\epsilon_{1p} - E_{1n}) (\epsilon_{1p} + E_{1n}) = (\epsilon_{2p} - E_{2n}) (\epsilon_{2p} + E_{2n}), \]  
we have
\[ (\epsilon_{1p} + \epsilon_{2p} - E_{n}) = \frac{(p^2 + \omega^2/n^2)}{(\epsilon_{1p} + E_{1n}) (\epsilon_{2p} + E_{2n})}. \]

If we use the last relation in Eq. (4), we arrive at the simplest relativistic generalization of the Coulomb wave function:
\[ \varphi_c' = \Omega_p \varphi_c, \quad \Omega_p = \frac{(\epsilon_{1p} + m_1) (\epsilon_{2p} + m_2)}{2 \mu (\epsilon_{1p} + \epsilon_{2p} + m_1 + m_2)}. \]

According to perturbation theory, the exchange of a transverse photon between the muon and the electron generates a correction to the fine shift of S levels,
\[ \Delta E_{nS} = \langle \varphi_{nS}', \langle K_T \rangle_{0F} \varphi_{nS} \rangle, \]  
where
\[ \langle K_T \rangle_{0F} = F^{-1} (G_0 K_T G_0)^+ F^{-1}, \]  
\[ K_T = \frac{\epsilon_1 \epsilon_2}{k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \gamma_{1i} \gamma_{2j}, \quad i, j = 1, 2, 3. \]

We use an integral representation of the Dirac delta function to perform the two-time operation in Eq. (3); as a result, we reduce integration with respect to the relative momenta \( p_0 \) and \( q_0 \) to contour integrals. In the ensuing calculations, it is convenient to employ the simplest recipes of the residue theory like
\[ \int_{-\infty}^{\infty} \frac{dp_0 e^{ip_0 t}}{(p_0 + E - \epsilon_p + i\epsilon)} (E - p_0 - \epsilon_p + i\epsilon) \]  
\[ = \frac{\pi i}{E - \epsilon_p} e^{-(\epsilon_p - E - i\epsilon)|t|}. \]

The matrix structure of the expression for \( K_T \) is simplified owing to the symmetry properties of the integrand. As a result, the fine shift as given by (9) can be expressed in terms of integrals with respect to 3-momenta. Specifically, we have
\[ \Delta E_{nS} = \frac{4\alpha^6 \mu^5}{(2\pi)^4} \int \frac{d^3 p N_p \Omega_p}{(p^2 + \omega^2)^2} \int \frac{d^3 q N_q \Omega_q}{(q^2 + \omega^2)^2} \]  
\[ \times \left\{ -(p + q)^2 \left( \frac{1}{M_{1p} + \frac{1}{M_{1q}}} \right) \left( \frac{1}{M_{2p}} + \frac{1}{M_{2q}} \right) \right. \]  
\[ + \frac{(p^2 - q^2)^2}{M_{1p} M_{1q} M_{2p} M_{2q}} \]  
\[ + 4 + 2(p^2 + q^2) \left( \frac{1}{M_{1p} \frac{1}{M_{1q}}} + \frac{1}{M_{2p} \frac{2q}} \right) \right\}, \]
where
\[ N_p = \sqrt{M_{1p} M_{2p}}, \quad M_{1p} = \epsilon_{1p} + m_1, \]  
\[ M_{2p} = \epsilon_{2p} + m_2. \]

The coefficients of the logarithmic terms, \( C_0 \) and \( C_2 \beta \), are a priori unknown. Therefore, it is natural to analyze first the contributions of order \( \alpha^6 \mu^5 / (m_1 m_2) \times \ln^2 \beta^{-1} \). Such corrections are associated with typical integrals referred to here as standard integrals:
\[ I_{st}^1 = \int_0^\infty \int_0^\infty \frac{dp}{p^2 + \gamma^2} \int_0^\infty \frac{dq}{q^2 + \gamma^2} \sqrt{p^2 + \beta^2} \sqrt{q^2 + \beta^2} \]  
\[ \beta \ll 1; \]  
\[ I_{st}^2 = \int_0^\infty \frac{dpp}{\sqrt{p^2 + \beta^2} (p^2 + \gamma^2)} \int_0^\infty \frac{dq}{q^2 + 1} \ln \left| \frac{p - q}{p + q} \right|. \]

The integral \( I_{st}^1 \) stems from terms proportional to \( (p^2 - q^2) / (p - q)^2 \) and is free from positive powers of \( p \) and \( q \) in the numerator. The integral \( I_{st}^2 \) arises in the terms of \( \Delta E_{nS} \) that include the Coulomb factor \( (p - q)^{-2} \). The calculation of \( C_2 \) yields
\[ C_2 = 1/\pi^2, \]  
so that the corresponding correction is
\[ \Delta E_{1S} = \frac{\alpha^6 \mu^3}{m_1 m_2 \pi^2} \left( \frac{\pi^2}{2} + \sqrt{2} + \ln \beta^{-1} \right) \ln \beta^{-1}, \]  
\[ \Delta E_{2S} = \frac{\alpha^6 \mu^3}{8m_1 m_2 \pi^2} \left( \frac{\pi^2}{2} + \sqrt{2} + \ln \beta^{-1} \right) \ln \beta^{-1}. \]