\[ \hat{sl}(2) \oplus \hat{sl}(2)/\hat{sl}(2) \] Coset Theory Is a Hamiltonian Reduction
of the \[ \hat{D}(2|1; \alpha) \] Superalgebra

A. M. Semikhatov* and B. L. Feigin**

*Tamn Theoretical Physics Department, Lebedev Institute of Physics, Russian Academy of Sciences, Leninskii pr. 53, Moscow, 117924 Russia
**Landau Institute for Theoretical Physics, Russian Academy of Sciences, ul. Kosygina 2, Moscow, 117334 Russia

Received April 19, 2001; in final form, May 31, 2001

It is shown that \[ \hat{sl}(2)_{k_1} \oplus \hat{sl}(2)_{k_2} \bigoplus \hat{sl}(2)_{k_1+k_2} \] coset theory is a quantum Hamiltonian reduction of the exceptional affine Lie superalgebra \[ \hat{D}(2|1); \alpha \]. In addition, the \( W \) algebra of this theory is the commutant of the \( \mathcal{U}_q\hat{D}(2|1); \alpha \) quantum group. © 2001 MAIK “Nauka/Interperiodica”.

PACS numbers: 02.20.Uw; 11.25.Hf

In this paper, it is found that (i) the well-known \[ \hat{sl}(2)_{k_1} \oplus \hat{sl}(2)_{k_2} \bigoplus \hat{sl}(2)_{k_1+k_2} \] coset models are constructed) and, more recently, in [2] (in view of the popularity of the (2) coset theory is invariant under permutations of the three levels \( k_1, k_2, \) and \( k_3 = -k_1 - k_2 - 4 \) and that the parameter \( \alpha \) is defined modulo a order-six group of discrete transformation; moreover, the level \( \kappa \) on the left-hand side of (1) can be chosen to be equal to the level of any of the three \( \hat{sl}(2) \) subalgebras in \( D(2|1); \alpha \), so that it is convenient to specify these three levels \( \kappa_1, \kappa_2, \) and \( \kappa_3 \), which are related by the equation \( 1/\kappa_1 + 1/\kappa_2 + 1/\kappa_3 = 0 \), instead of the parameter \( \alpha \) and the level \( \kappa \). The relations \( \kappa_1 = 1/(k_1 + 2), \kappa_2 = 1/(k_2 + 2), \) and \( \kappa_3 = -1/(k_1 + k_2 + 4) \) then hold; modulo the aforementioned arbitrariness, we also have \( \alpha = -1 - (k_1 + 2)/(k_1 + 2) \).

In addition to (1), we will show that the algebra of local fields of the coset theory in question coincides with the \( W \) algebra \( \hat{W}D(2|1); \alpha \) defined by the root system of the \( D(2|1); \alpha \) Lie superalgebra; that is,

\[ \hat{sl}(2)_{k_1} \oplus \hat{sl}(2)_{k_2} \bigoplus \hat{sl}(2)_{k_1+k_2} = \hat{W}D(2|1); \alpha. \] (2)

The equality in (1) requires describing the relation between the parameters on the left- and on the right-hand side. For this, we recall that the definition of a coset theory is invariant under permutations of the three levels \( k_1, k_2, \) and \( k_3 = -k_1 - k_2 - 4 \) and that the parameter \( \alpha \) is defined modulo a order-six group of discrete transformation; moreover, the level \( \kappa \) on the left-hand side of (1) can be chosen to be equal to the level of any of the three \( \hat{sl}(2) \) subalgebras in \( D(2|1); \alpha \), so that it is convenient to specify these three levels \( \kappa_1, \kappa_2, \) and \( \kappa_3 \), which are related by the equation \( 1/\kappa_1 + 1/\kappa_2 + 1/\kappa_3 = 0 \), instead of the parameter \( \alpha \) and the level \( \kappa \). The relations \( \kappa_1 = 1/(k_1 + 2), \kappa_2 = 1/(k_2 + 2), \) and \( \kappa_3 = -1/(k_1 + k_2 + 4) \) then hold; modulo the aforementioned arbitrariness, we also have \( \alpha = -1 - (k_1 + 2)/(k_1 + 2) \).

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charge of the resulting \( W \) algebra depends on the parameter \( \kappa \) introduced here). For the Lie superalgebra \( \mathfrak{g} \) such that all its odd roots are isotropic [in particular, for \( D(2|1; \alpha) \)], we define \( \mathcal{W} \mathfrak{g} \) in a way similar to that in the bosonic case, the only distinction being that the operator corresponding to each odd root \( \alpha \), is replaced by \( \hat{f} e^a \cdot \mathfrak{g} \), where \( a \) satisfy the condition \( a, a_i = 1 \) and where the set of all screenings satisfies the nilpotent subalgebra in \( \mathcal{U}_q \mathfrak{g} \). Operators of the form \( \hat{f} e^a \cdot \mathfrak{g} \), where \( a_i = 1 \), will be referred to as fermionic screenings, while all the remaining ones are called bosonic screenings.

Described below are basic steps leading to (1) and (2).

1. **Reminder: \( D(2|1; \alpha) \) superalgebra** [3]. The \( D(2|1; \alpha) \) superalgebra of dimension \( 9|8 \) has the even part \( \mathfrak{s}l(2) \oplus \mathfrak{s}l(2) \oplus \mathfrak{s}l(2) \), whose action on the odd part is the product of two-dimensional representations. This algebra admits a set of roots where all three simple roots are fermionic; the Chevalley generators \( \psi_i, i = 1, 2, 3 \), satisfy the relations \( [\psi_i, \psi_j] = 0, [\psi_2, \psi_3] = 0, [\psi_3, \psi_2] = 0 \) (here, \([\ , \] \) stands for a supercommutator), and \( [\psi_2, [\psi_1, \psi_3]] + (\alpha + 1)[\psi_3, [\psi_1, \psi_2]] = 0 \). Therefore, the nilpotent subalgebra also contains three even elements

\[
e^{(1)} = \frac{K_1}{2} [\psi_2, \psi_3], \quad e^{(2)} = \frac{K_2}{2} [\psi_1, \psi_3],
\]

\[
e^{(3)} = \frac{K_3}{2} [\psi_1, \psi_2],
\]

which are upper-triangle generators of three \( \mathfrak{s}l(2) \) subalgebras, and, in addition, one odd element \( \psi_0 = [\psi_1, e^{(1)}] = [\psi_2, e^{(2)}] = [\psi_3, e^{(3)}] \). Here, \( 1/K_1 + 1/K_2 + 1/K_3 = 0 \), while the parameter \( \alpha \) is then defined as \( \alpha = -1 - (K_1/K_2/K_3) \).

2. **Hamiltonian reduction of \( \hat{D}(2|1; \alpha) \).** The Hamiltonian reduction of superalgebras may require introducing auxiliary fields used in imposing constraints. For the \( \hat{D}(2|1; \alpha) \) algebra, there are a few natural options of such fields and constraints. The scheme that we consider is asymmetric in three fermionic roots—namely, we introduce a free fermionic system generated by \( \eta \) and \( \xi \) and characterized by the operator product \( \eta(z)\xi(w) = 1/(z - w) \) and impose the constraints \( \psi_1(z) = \eta(z), \psi_2(z) = \eta(z), \) and \( \psi_3(z) = \xi(z) \), and, accordingly, \( e^{(1)}(z) = -\kappa_1/2, e^{(2)}(z) = -\kappa_2/2, e^{(3)}(z) = 0, \) and \( \psi_0(z) = 0 \). In order to construct the relevant Becchi–Rouet–Stora–Tyutin (BRST) differential, we introduce ghosts—that is, first-order bosonic and fermionic systems featuring the relevant operator products \( \beta_j(z)\eta(w) = -\delta_j/(z - w) \) and \( B_j(z)C_j(w) = \delta_j/(z - w) \). The BRST differential is then given by \( \mathcal{Q} = \hat{f} \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \mathcal{F}^{(2)} \), where

\[
\mathcal{F}^{(0)} = \psi_1\psi_2 + \psi_2\psi_3 + e^{(1)}C_1 + e^{(2)}C_2 + e^{(3)}C_3 + \psi_0\psi_0 - \frac{2}{\kappa_1}B_1B_2 - \frac{2}{\kappa_2}B_2B_3C_3 - \frac{2}{\kappa_3}B_3B_1, \\
\mathcal{F}^{(1)} = -\gamma_1\gamma_2 - \gamma_2\gamma_3 - \xi_3\eta, \\
\mathcal{F}^{(2)} = \frac{\kappa_1}{2}C_1 + \frac{\kappa_3}{2}C_2.
\]

The cohomology of this BRST operator is the result of the Hamiltonian reduction in question. More precisely, the cohomology of \( \mathcal{Q} \) inevitably contains the Heisenberg algebra, since we have introduced the auxiliary fields \( \eta \) and \( \xi \)—our objective is to find, in a cohomology with zero ghost number, a \( W \) algebra that commutes with this Heisenberg algebra \( \mathcal{H}_0 \). One can easily find a current that generates the \( \mathcal{H}_0 \) algebra. The result is

\[
\tilde{H} = 2h(3) + 2B_3C_3 + \beta_1\gamma_0 + \beta_2\gamma_3 - \beta_3\gamma_1 + \eta\xi,
\]

where \( h^{(i)} \) are the Cartan currents of three \( \mathfrak{s}l(2) \) subalgebras in \( \hat{D}(2|1; \alpha) \). We further note that the cohomology of \( \mathcal{Q} \) also contain the family of stress–energy tensors that depends on the parameter \( j \) and which has the form

\[
\tilde{T}(j) = \tilde{T} + \partial_1B_1 + \partial_2B_2 + (2j - 3)B_3\partial_3 \\
+ 2(j - 1)\partial_3B_3 + (j - 1)\beta_0\gamma_0 + j\beta_0\gamma_0 \\
+ (j - 2)\beta_1\gamma_1 + (j - 1)\beta_2\gamma_2 + (j - 2)\beta_3\gamma_3 \\
+ (j - 1)\partial_2B_2(1 - j)\beta_3\gamma_3 + (2 - j)\beta_3\gamma_3 \\
+ (j - 2)\eta\partial_3 + (j - 1)\partial_3\xi + \partial h^{(1)} + \partial h^{(2)} + (2j - 1)\partial h^{(3)}.
\]

The only combination that commutes with \( \tilde{H} \) and which is independent of \( j \) is given by

\[
\tilde{T} = \tilde{T}(j) - \frac{1}{2(1 + 2\kappa_3)}\tilde{H} - \frac{4(j - 1)\kappa_3 + 2j - 3}{2(1 + 2\kappa_3)}\partial H.
\]

The central charge of this stress–energy tensor can be represented as

\[
\hat{c} = \frac{3(1 - 2\kappa_1)(1 - 2\kappa_2)}{1 + 2\kappa_3}.
\]