Polarization States in Quantum Optics

Unpolarized Light in Quantum Optics

J. Söderholm*, G. Björk*, and A. Trifonov**

*Department of Electronics, Royal Institute of Technology (KTH), Electrum 229, SE-164 40, Kista, Sweden
e-mail: jonas@ele.kth.se
**Ioffe Physical Technical Institute, St. Petersburg, 194021 Russia

Received November 17, 2000

Abstract—We present a new derivation of the unpolarized quantum states of light, whose general form was first derived by Prakash and Chandra [6]. Our derivation makes use of some basic group theory, is straightforward, and offers some new insights. © 2001 MAIK “Nauka/Interperiodica”.

INTRODUCTION

The quantum theory of polarization formally has the same structure as that for relative phase. This follows from the fact that both observables measure the relative properties of two bosonic modes. As is well-known, the phase shift is generated by one of the Schwinger bosonic SU(2) operators [1], which form the Lie algebra for linear lossless transformations of two bosonic modes [2]. Similarly, when considering polarization, another SU(2) operator appears generating geometric rotations. This makes it possible to treat the polarization in the same manner as was successfully applied to the relative phase [3–5], that is, considering the different excitation manifolds separately. In this way, we are able to easily find all the unpolarized quantum states of light first derived by Prakash and Chandra [6], and later treated by Agarwal [7] and by Lehner, Leonhardt, and Paul [8]. We think our treatment offers some insight into the nature of polarization at a quantum level.

THE SCHWINGER BOSONIC SU(2) OPERATORS

We will consider a transverse field of light (or more generally, a transverse field of bosons) that can be decomposed into two linearly polarized modes, called $a$ and $b$. The field is supposed to have a well-defined direction of propagation. Assuming that the two modes have the same frequency, any lossless coupling will leave the number of excitations (photons) unchanged. It is therefore natural to introduce the Schwinger realization of the SU(2) algebra [1]

\[
\hat{L}_1 = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger), \quad \hat{L}_2 = \frac{1}{2i}(\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger), \quad \hat{L}_3 = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}),
\]

where $\hat{a}$ and $\hat{a}^\dagger$ denote the annihilation and creation operators of mode $a$, respectively. The SU(2) operators satisfy

\[
[\hat{L}_k, \hat{L}_l] = i\epsilon_{klm} \hat{L}_m,
\]

where the Levi–Civită tensor $\epsilon_{klm}$ is equal to 1 and $-1$ for even and odd permutations of its indices, respectively, and zero otherwise. The Casimir operator is

\[
\hat{\mathcal{L}}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 = \frac{\hat{N} - \hat{N}}{2} + 1,
\]

where $\hat{\mathcal{L}}^2$ is equal to 1 and $\hat{b}^\dagger \hat{b}$ is the total photon number. Thus, $\hat{\mathcal{L}}^2$ commutes with all the generators $\hat{L}_k$, $k = 1, 2, 3$, which shows that the coupling described by them are indeed lossless. The Schwinger operators $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_3$, and $\hat{L}_4$ are proportional to the Stokes operators $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$, and $\hat{S}_0$, that can be constructed by quantizing the Stokes parameters used to characterize the polarization state of classical light fields.

The evolutions under the SU(2) algebra span all the linear lossless transformations of two bosonic modes [2], and they can be written as

\[
\hat{U}(\phi_1, \phi_2, \phi_3) = e^{i(\phi_1 \hat{L}_1 + \phi_2 \hat{L}_2 + \phi_3 \hat{L}_3)},
\]

where $\phi_1$, $\phi_2$, and $\phi_3$ are real. In physical terms, any combination of differential phase shifts and rotations around the direction of propagation can be expressed in this form using the appropriate variables $\phi_k$. Since all the generators commute with the total photon number, any evolution takes the form of a direct sum

\[
\hat{U}(\phi_1, \phi_2, \phi_3) = \bigoplus_{N=1}^{\infty} \hat{U}_N(\phi_1, \phi_2, \phi_3),
\]
where $\hat{U}_N$ operates on the $N$-dimensional Hilbert space $\mathcal{H}_N$ formed by all the two-mode states with $N-1$ photons. This means that the matrix corresponding to $\hat{U} (\phi_1, \phi_2, \phi_3)$ is block-diagonal if the basis is chosen to be the two-mode number states ordered by their excitation manifold, for example, $(|0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |0, 2\rangle, |1, 1\rangle, |2, 0\rangle, |0, 3\rangle, \ldots)$. Similarly, the Hilbert space of two harmonic oscillators can be written

$$\mathcal{H}_\infty \otimes \mathcal{H}_\infty = \bigoplus_{N=1}^{\infty} \mathcal{H}_N. \quad (8)$$

**UNPOLARIZED QUANTUM STATES**

Prakash and Chandra argued that the unpolarized quantum states should be defined as those states that are invariant under any geometric rotation and any phase shift. From these conditions the most general form of an unpolarized quantum state was derived [6]. An equivalent definition, but using different derivations, led Agarwal, and later Lehner et al., to arrive at the same result [7, 8]. Here we present a fourth, alternative, more straightforward, derivation which makes use of Shur’s lemma.

In mathematical form, the invariance requirements can be written

$$[\hat{\rho}, e^{i\theta_k \hat{L}_k}] = 0, \quad \forall \theta_k, \quad k = 2, 3, \quad (9)$$

where $\hat{\rho}$ is the density operator of the quantum state. Since this is valid for all $\theta_2$ and $\theta_3$, we have

$$[\hat{\rho}, \hat{L}_k] = 0, \quad k = 2, 3. \quad (10)$$

We start by observing that $\hat{L}_1$ can be expressed in $\hat{L}_2$ and $\hat{L}_3$, as is seen from Eq. (4). Thus, any operator that commutes with both $\hat{L}_2$ and $\hat{L}_3$, also commutes with $\hat{L}_1$. Therefore, the unpolarized states satisfy

$$[\hat{\rho}, \hat{L}_k] = 0, \quad k = 1, 2, 3. \quad (11)$$

This implies, according to Shur’s lemma [9], that the density operator is a multiple of the identity operator in the respective Hilbert spaces $\mathcal{H}_N$. Since density operators are Hermitian, we have

$$\hat{\rho} = \bigoplus_{N=1}^{\infty} r_N \hat{1}_N, \quad (12)$$

where $\hat{1}_N$ is the identity operator in $\mathcal{H}_N$, and all $r_N$ are real and nonnegative. Writing this equation explicitly gives

$$\hat{\rho} = r_1|0, 0\rangle\langle 0, 0| + r_2(|0, 1\rangle\langle 0, 1| + |1, 0\rangle\langle 1, 0|) + \ldots \quad (13)$$

Furthermore, the trace of any density operator equals unity, so we find

$$\sum_{N=1}^{\infty} N r_N = 1. \quad (14)$$

We note that the vacuum state is the only pure state that is unpolarized, and that the unpolarized mixed states are totally mixed in each manifold. Any two-mode thermal state is hence unpolarized.

It is clear that Eq. (10) is equivalent to

$$[\hat{\rho}, \hat{U} (\phi_1, \phi_2, \phi_3)] = 0, \quad \forall \phi_1, \phi_2, \phi_3, \quad (15)$$

where $\hat{U} (\phi_1, \phi_2, \phi_3)$ denotes the linear lossless transformations (6). Moreover, the form of the unpolarized states (12) makes them invariant under any unitary, not necessarily linear, lossless transformation.

**CONCLUSIONS**

We have investigated the quantum mechanical definition of unpolarized light (10), suggested by Prakash and Chandra [6]. (The choice $k = 1, 2,$ or $k = 1, 3,$ leads to equivalent definitions.) The suggested definition is very concise and can experimentally be stated as: an unpolarized state is invariant under any rotation around its direction of propagation and any differential phase shift, or any combination thereof. Note that this definition is more severe than the classical definition of unpolarized light, as pointed out by Agarwal [7]. Classically, the term “unpolarized” only implies invariance of the second order moments of the two modes under differential phase shifts and geometrical rotations. The quantum mechanical definition implies invariance of the moments of all orders.

It is somewhat surprising that such a relatively "mild" defining condition (10) leads to such severe restraint on the density matrix as Eq. (12). If one defines the circularly polarized states as the states that are invariant under rotation around the direction of propagation, one finds that there are $N + 1$ such (orthonormal) states in every excitation manifold $N$. These states can be used to form a Hermitian circular polarization operator in a similar manner as the relative phase operator has been constructed [3, 4]. In the $N$th excitation manifold, one can subsequently construct up to $N$ additional, complementary, operators [10]. An alternative definition of an unpolarized transverse state would be that such a state should be invariant under all transformations generated by these complementary operators. A direct check of what requirement on the density matrix this alternative (but in our eyes seemingly stronger) definition of unpolarized light dictates, again leads to Eq. (12). Since Eq. (10) is commendably concise, and is experimentally testable using only linear components (only variable phase-plates are needed), we think it is a correct and convenient mathematical definition of unpolarized light.