On a Simple Model of the Photonic or Phononic Crystal

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A model is proposed for a one-dimensional dielectric or elastic superlattice (SL) that relatively simply describes the frequency spectrum of electromagnetic or acoustic waves. The band frequency spectrum is reduced to minibands contracting with increasing frequency. A procedure is suggested for obtaining local states near a defect in a SL, and the simplest of these states is described. Conditions for the initiation of Bloch oscillations of a wave packet in a SR are discussed. © 2001 MAIK “Nauka/Interperiodica”.

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1. By a photonic crystal is meant a macroscopic periodic structure composed of two spatially alternating dielectrics differing in dielectric constants (velocities of electromagnetic waves) [1]. Analogously, by a phononic crystal is meant a periodic structure composed of two alternating elastic materials differing in elastic moduli and velocities of sound (the general acoustics theory of layered media is expounded in [2], and a useful bibliography on acoustic SLs is given in one of the last publications [3]). A great number of publications are devoted to studying the frequency spectrum of SLs. It is clear that, in the general case, this spectrum is extremely complicated and contains a system consisting of both a great number of eigenfrequency bands and gaps corresponding to forbidden frequencies of eigenmodes. In order to characterize such spectra qualitatively and to illustrate their main quantitative features, it would be appropriate to use simple models that allow for these features. The well-known 1D Kronig–Penney model [4] may serve as an example of such a model in the electronic theory of crystals. In this work, a model of a SL is proposed that provides an analytical description of the high-frequency part of its spectrum and suggests a possible implementation of an interesting acoustic SL.

Consider a SL in the form of alternating plane-parallel layers of two materials differing in either elastic or dielectric (depending on the implementation of interest) characteristics. Denote the layer thicknesses by $d_1$ and $d_2$; then, the SL period equals $d = d_1 + d_2$. The elastic or electromagnetic field inside each material, which is assumed to be isotropic, is described by the wave equation

$$\frac{\partial^2 u^\alpha}{\partial t^2} - c^2_\alpha \frac{\partial^2 u^\alpha}{\partial x^2} = 0, \quad \alpha = 1, 2,$$

where $c_\alpha$ is the wave velocity in the layer of the $\alpha$ type.

The velocity of light in a dielectric equals $c_\alpha = c / \sqrt{\varepsilon_\alpha}$ ($c$ is the velocity of light in free space), and that in an elastic medium equals $c_\alpha = \sqrt{\mu_\alpha / \rho_\alpha}$; $\varepsilon_\alpha$, $\mu_\alpha$, and $\rho_\alpha$ ($\alpha = 1, 2$) are dielectric constants, elastic moduli, and mass densities, respectively.

Consider a wave propagating along the $X$ axis perpendicular to the layers. In this case, waves of two possible polarizations do not interact, and it is possible to study scalar fields $u^{(\alpha)} (\alpha = 1, 2)$.

The standard boundary conditions will be formulated as applied to the acoustic problem. The displacements $u^{(\alpha)}$ and stresses $\sigma^{(\alpha)} = \mu^{(\alpha)} (\partial u^{(\alpha)}) / \partial x$ at the layer boundaries will be considered continuous. It is known that, by virtue of the periodicity of a structure with a period of $d$, eigenmodes can be characterized by a quasi-wave number $k$, considering that the field in a unit cell with the number $n$ takes the form

$$u_n(x) = u_0(x - n d) e^{i k n d}.$$  

The dispersion equations in this problem were obtained by Rytov for both the electromagnetic field [5] and acoustics [6]

$$\cos k d = \cos k_1 d_1 \cos k_2 d_2 - \frac{1}{2} \left( k_2 + \frac{k_2}{k_1} \right) \sin k_1 d_1 \sin k_2 d_2,$$

where $k_1 = \omega / c_1$ and $k_2 = \omega / c_2$ ($\omega$ is frequency). Equation (3) determines the frequency as an implicit function of the quasi-wave number. It allows the spectrum of long-wavelength vibrations ($kd \ll 1$) to be described readily, for which a sound spectrum with averaged elas-

1Because I am interested mainly in narrow frequency bands, the frequency dispersion of $\varepsilon$ can be neglected, and $\varepsilon$ can be related to the corresponding frequencies.
The relationship, for example, as a possible implementation of such a system, together with thin vacuum interlayers may serve, for instance, in solids [7] or planar defects in crystals [8]. If the layer determined by the stresses at the joint is small, the system at hand is reduced to a more general case, i.e., the following boundary conditions are fulfilled at their joints: (1) continuity of the normal stresses $\sigma$; (2) occurrence of a jump of displacements at a soft interlayer determined by the stresses at the joint

$$[u]^+ = M\sigma \equiv \mu_1 M \left( \frac{\partial u}{\partial x} \right),$$

where $M = P(\rho_1/\rho_2)$. A set of such boundary conditions at a fixed $M$ is used in describing capillary phenomena in solids [7] or planar defects in crystals [8]. If the parameter $M$ is small, the system at hand is reduced to a periodic sequence of elastic sections weakly bound together. A chain of piezoelectric sections bound together by thin vacuum interlayers may serve, for example, as a possible implementation of such a system. Then, the coupling of elastic vibrations in neighboring sections would be accomplished through electromagnetic oscillations in vacuum gaps.

To illustrate the distribution of the roots $\omega = \omega(k)$ of Eq. (5), this equation will be represented in the form

$$\cos k d = \cos z - Q \sin z,$$

where $z = k d/\omega d/\omega_1$ and $Q = P(\rho_1 M/2\rho_1 d)$. Consider the graphical construction in Fig. 1. The figure shows a plot of the right-hand side of Eq. (7). When it runs over values between ±1, the roots of the equation run over values within intervals marked off on the abscissa axis.

Note that the allowed frequencies are localized in contracting intervals at values $k d = \pm m \pi$, where $m$ is a large integer, as $z$ increases. Under the condition that $m^2 Q \gg 1$, the dispersion laws in these intervals take the form

$$\omega = m \omega_0 + \frac{2 \Omega}{m} \left( \frac{\sin^2 (kd/2)}{\cos^2 (kd/2)} \right), \quad m = 2p + 1;$$

where $\omega_0 = \pi c/d$ and $\Omega = c/\pi Q d$. It is clear that Eq. (8) gives the size quantization phonon spectrum in a layer of thickness $d$, whose levels are split into minibands because of low “transparency” of interlayer boundaries. An attempt to analyze the character of the SL band spectrum was made in [9], where the dispersion relation (Eq. (3)) was derived once again. However, their analysis is not satisfactory in a limiting case close to that considered in this work, because it leads to the conclusion that the miniband widths do not vary with increasing frequency.

Consider Eq. (8) from another point of view: Eq. (8) describes the spectrum of a pseudo-quantum particle for which the Schrödinger equation within the tight-binding model takes the form (for $m = 2p + 1$)

$$i \frac{\partial \psi_n}{\partial t} = m \omega_0 \psi_n - \frac{\Omega}{m} (2 \psi_n - \psi_{n+1} - \psi_{n-1}),$$

or (for $m = 2p$)

$$i \frac{\partial \psi_n}{\partial t} = m \omega_n \psi_n + \frac{\Omega}{2m} (2 \psi_n + \psi_{n+1} + \psi_{n-1}).$$

Actually, Eqs. (9) and (9a) are equations for the envelope curve of SL vibrations taken at discrete points (at joints). As usual, the order of the derivative with respect to time decreases in such equations. These equations describe analytically the dynamics of a wave packet corresponding to the allowable high frequencies. Using the explicit form of the dispersion laws (Eq. (6)) and simple Eqs. (7) and (8), the passage of wave packets through the system under study can be described readily, and explicit relationships can be proposed for comparison with possible experimental results.

\footnote{A more general case, i.e., $d_2 \rightarrow 0$ and $c_2 \rightarrow 0$ at $d_3/c_3 = \text{const}$, could be considered; however, no new results arise in this case.}

\footnote{The contraction of bands with increasing frequency was also noted previously; in particular, this was mentioned in [3].}