As is known [1], high-frequency acoustic vibrations can arise in heated ducts with a supercritical liquid. These vibrations cause pulsations of the flow rate of the liquid, cyclic stresses in the device structure, and even the destruction of the latter. For example, experiments [2] showed that pressure oscillations with a frequency up to $10000 \text{s}^{-1}$ and amplitude up to 27 atm arise in supercritical liquid hydrocarbons in a turbulent flow regime as they are heated in a duct (with a liquid-propellant-engine cooling system) at a steady-state pressure of ~35 atm. These oscillations were responsible for the formation of cracks and flaws in the experimental duct. Therefore, the excitation of acoustic vibrations by heating a supercritical liquid flowing in a duct is a problem of current interest.

There is extensive experimental information concerning the origination of thermal acoustic vibrations of a supercritical heat-transfer liquid [3–6]. The phenomenon was explained in [1, 7] as follows. When heat is supplied from walls, the field of the turbulent flow of a supercritical liquid can be divided into the turbulent core and viscous sublayer near the walls. The viscous sublayer is characterized by a steep temperature profile, while the profile in the turbulent core is much smoother because of turbulent mixing. In addition, the viscous liquid sublayer is more compressible than the flow core. Therefore, the viscous sublayer provides the phase relations between variations of pressure and heat flux (feedback) that are necessary for the excitation of acoustic vibrations. This mechanism was used in [7] to numerically investigate the excitation of acoustic vibrations in a heated supercritical liquid.

In this paper, the excitation of longitudinal acoustic vibrations by heating a supercritical liquid flowing in a duct of length $l$ and radius $r_0$ is investigated theoretically. It is assumed that heat supply to the flowing liquid is distributed along the duct length, whereas the heat-flux density $q$ for the steady-state process is constant. Similar to [1, 7], quasi-static relations are used to describe heat exchange between the liquid vibrating with acoustic frequency and the duct walls. In addition, as is accepted in acoustics [8], the supercritical liquid is considered as inviscid.

Under these assumptions, equations of continuity, motion, and energy take the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \quad (1)$$
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial \rho}{\partial x}, \quad (2)$$
$$\rho \left( \frac{\partial i}{\partial t} + u \frac{\partial i}{\partial x} \right) = \frac{1}{r} \frac{\partial r}{\partial r} (qr) + \frac{\partial p}{\partial t}, \quad (3)$$

where $i$ is the specific enthalpy of the liquid. The other notation is conventional. The liquid density is treated as a function of thermodynamic parameters; i.e.,

$$\rho = \rho(p, s). \quad (4)$$

The linearization of Eqs. (1)–(3) yields the following approximate equations:

$$\frac{\partial \rho'}{\partial t} + \rho \frac{\partial u'}{\partial x} + u \frac{\partial \rho'}{\partial x} = 0, \quad (5)$$
$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho} \frac{\partial \rho'}{\partial x}, \quad (6)$$
$$\rho \left( \frac{\partial i'}{\partial t} + u \frac{\partial i'}{\partial x} \right) = \frac{1}{r} \frac{\partial r}{\partial r} (qr') + \frac{\partial p'}{\partial t}. \quad (7)$$

Hereafter, perturbation is marked by a prime.

Approximate equations (5)–(7) can be used for investigating the excitation of acoustic vibrations in a flowing supercritical liquid at $Sh \gg 1$, where $Sh = \frac{\omega l}{u}$ is the Strouhal number for acoustic vibrations and $\omega$ is the cyclic frequency. To linearize Eq. (4), one should have the density of a supercritical liquid $\rho$ as a function
of the enthalpy \( \dot{i} \) at constant pressure. Here, we use the empirical dependence

\[
\rho = \left[ \rho_0^{-1} + b(i - i_0) \right]^{-1}, \quad b = \text{const},
\]

proposed in [9] for helium, where \( \rho_0 \) and \( i_0 \) are the density and enthalpy of the supercritical liquid at the duct entrance.

With insignificant error, this dependence can be used as well for other supercritical liquids. For example, for supercritical water vapor, it is accurate to not worse than \( \sim 20\% \). In view of the last expression, the linearization of (4) yields

\[
\frac{\rho}{\rho_c} = \frac{p'}{p_c} - bp\dot{i}, \quad (8)
\]

where \( c_s \) is the adiabatic speed of sound.

Substituting Eq. (8) into Eq. (5) and taking into account both Eq. (7) and the equality

\[
\left( \frac{1}{\rho c_s} - b \right) \frac{\partial p'}{\partial t} = \frac{b}{r \partial r} (q' r) - \frac{\partial u'}{\partial x}, \quad (9)
\]

we obtain the equation

\[
b p \frac{\partial \dot{i}}{\partial x} = - \frac{1}{\rho} \frac{\partial \rho}{\partial x}.
\]

Differentiating Eqs. (6) and (9) with respect to \( x \) and \( t \), respectively, and eliminating \( u' \), we derive the equation

\[
\left( \frac{1}{\rho c_s} - b \right) \frac{\partial^2 p'}{\partial t^2} = \frac{1}{\rho} \frac{\partial p'}{\partial x} + \frac{b}{r \partial r} \left( \frac{\partial q'}{\partial r} \right). \quad (10)
\]

To be solved, this equation requires an expression for \( q' \), which, in general, depends on heat transfer in both the liquid and the duct wall. As is known [7], the quantity \( q' \) is primarily determined by the variation of viscous-sublayer thickness under acoustic vibrations. Therefore, using the condition of mass conservation in the viscous sublayer, one can derive the relation

\[
\frac{q'}{q} = \frac{p'}{p} = \frac{1}{\gamma} \frac{p'}{p}, \quad (11)
\]

where \( \gamma \) is the specific-heat ratio.

The last formula is derived with allowance for the fact that, at high-frequency vibrations in a duct, a state of the liquid can be described in the quasi-adiabatic approximation [8]. For a steady-state flow of the supercritical liquid, enthalpy is distributed linearly along the duct [10]:

\[
\dot{i} = i_0 + \frac{q_l x}{\rho_0 u_0 r_0},
\]

where \( q_l \) is the heat-flux density on the duct surface. Averaged along the duct length, the last expression yields

\[
\langle \dot{i} \rangle = i_0 + \frac{q_l l}{\rho_0 u_0 r_0}. \quad (12)
\]

For a turbulent flow, liquid velocity at each point of a duct cross-section is close to its average value over this section [10]. Then, the heat flux is a linear function of the radius, so that

\[
\frac{q}{q_l} = \frac{r}{r_0}. \quad (13)
\]

In this case, with allowance for Eqs. (11)–(13), Eq. (10) can be written as

\[
\frac{\partial^2 (p')}{\partial t^2} = \alpha^2 \frac{\partial^2 (p')}{\partial x^2} + 2\delta \frac{\partial (p')}{\partial t}; \quad (14)
\]

where \( \alpha^2 = \frac{\langle c_s^2 \rangle}{1 - \gamma \beta p_0} \) and \( \delta = \frac{b}{1 - \gamma \beta p_0 r_0} \).

Equation (14) should be complemented by boundary conditions corresponding to the properties of the device in whose duct the excitation of acoustic vibrations is investigated. In this paper, we consider a duct with acoustically open ends. Then, the boundary conditions have the form

\[
p' = 0 \text{ at } x = 0,
\]

\[
p' = 0 \text{ at } x = l.
\]

Solving Eq. (14) with these boundary conditions by the method of separation of variables, we obtain the expression

\[
\frac{p'}{p} = e^{\delta x} \sum_{k=1}^{\infty} B_k \sin \frac{\pi k x}{l} \cos \left( \sqrt{\omega_k^2 - \delta^2} t + \varphi_k \right), \quad (15)
\]

where \( k = 1, 2, 3 \ldots \).

The quantities \( B_k \) and \( \varphi_k \) are determined from the initial conditions, while the natural frequency

\[
\omega = \frac{\pi k \alpha}{l}
\]

is found from the boundary conditions. Expression (15) shows that, at \( \delta > 0 \), the external heating of the supercritical liquid flowing in the duct excites acoustic vibrations with the increment

\[
\mu = \frac{2\pi \delta}{\sqrt{\omega_k^2 - \delta^2}}.
\]

According to Eq. (14), the excitation of acoustic vibrations ceases at the pressure \( p \geq \frac{1}{\gamma \beta} \), which is usually much higher (by about five or six times) than the critical pressure. Therefore, in a supercritical liquid, the excita-