1. INTRODUCTION

The properties of a Bose–Einstein condensate in which particle motion is “frozen” or reduced to zero-point oscillations in one or two directions are the subject of intensive studies [1–14]. In experiments, cigar-shaped quasi-one-dimensional condensates are created by using optical dipole traps [1]. A quasi-two-dimensional condensate was created in an array of disc-shaped traps provided by the periodic potential of a laser beam [2]. When the traps are sufficiently deep, the motion along the array is frozen and the condensate splits into several independent condensates confined in separate potential wells.

Important experimental information about the properties of a Bose–Einstein condensate confined in a three-dimensional trap can be extracted by measuring the time-dependent density of the expanding atomic cloud after the trapping potential is switched off. In the mean-field approximation, the dynamics of a dilute condensate is described by the Gross–Pitaevskii equation [14]

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V_{\text{ext}}(r)\psi + g|\psi|^2\psi, \tag{1} \]

where

\[ V_{\text{ext}}(r) = \frac{1}{2} m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \]

is the trapping potential,

\[ g = 4\pi\hbar^2 a_s/m \tag{2} \]

is the nonlinear coupling constant associated with an atom–atom scattering length \( a_s \), and the condensate wave function \( \psi \) is normalized to the number of atoms

\[ \int |\psi|^2 \, d\mathbf{r} = N. \tag{3} \]

If the number of atoms is sufficiently large, then the Gross–Pitaevskii equation can be transformed into hydrodynamic equations that admit simple self-similar solutions describing both oscillations of a gas in a parabolic trapping potential and its free three-dimensional expansion after the potential is switched off [15–18]. This theory is perfectly consistent with experiment.

A different situation arises when some degrees of freedom of the expanding condensate remain frozen. Recently, condensate expansion was investigated in quasi-one-dimensional waveguides [1] and in systems of two-dimensional discs [2]. This promising line of research was pursued in several studies. In [19], quasi-one-dimensional condensate expansion was analyzed without taking into account the transverse “quantum pressure.” In [20], the effects due to quantum pressure were taken into account for steady states, in which case only the two transverse modes contribute to the pressure. In [13], the ground states of condensates confined in cigar- and disc-shaped traps were calculated by a variational method, but no analysis of the dynamics of condensate expansion was presented.

In this paper, an analytical study of quasi-one-dimensional and quasi-two-dimensional condensate expansion is presented. Conditions are formulated under which the three-dimensional Gross–Pitaevskii equation can be reduced to analogous equations in fewer coordinates. These equations are solved in the hydrodynamic approximation under initial conditions corresponding to a trapped condensate in equilibrium before the trap is switched off. The condensate expands either along the axis of a quasi-one-dimensional waveguide or in the plane of a quasi-two-dimensional trap. However, if the conditions for reduction to Gross–Pitaevskii equations of lower dimension are violated, then the gas flow is three-dimensional. Three-dimensional effects in the flow are calculated by a variational
method. Finally, it is shown that the theoretical results agree with experiment.

2. QUASI-ONE-DIMENSIONAL AND QUASI-TWO-DIMENSIONAL CONDENSATE EXPANSION WITHOUT THREE-DIMENSIONAL EFFECTS

It is well known that the Gross–Pitaevskii equation can be formulated as a principle of least action with the action functional

$$ S = \int L dt, \quad L = \int \mathcal{L} dr, \quad (4) $$

where the Lagrangian density is

$$ \mathcal{L} = \frac{i}{2} (\psi^* \psi - \psi \psi^*) + \frac{\hbar^2}{2m} |\nabla \psi|^2 + V_{\text{ext}} |\psi|^2 + \frac{1}{2} g |\psi|^4. \quad (5) $$

In the case of a cigar- or disc-shaped trap, one can readily find conditions under which the tightly restrained degrees of freedom are frozen and the Gross–Pitaevskii equation reduces to a one- or two-dimensional equation, respectively. Even though this problem has been considered more than once, we briefly review here the basic points of the derivation in order to identify the essential parameters of the theory and formulate conditions for its applicability.

2.1. One-Dimensional Expansion

If the longitudinal frequency $\omega_\parallel$ for an axially symmetric trap is much less than the transverse trap frequency $\omega_\perp$,

$$ \lambda = \omega_\parallel / \omega_\perp \ll 1, \quad (6) $$

and the transverse zero-point energy is much higher than the nonlinear interaction energy per atom, then the transverse motion reduces to the ground state of particle oscillation, with the amplitude

$$ a_\perp = (\hbar / m \omega_\parallel)^{1/2}. \quad (7) $$

Denoting by $Z_0$ the characteristic size of the condensate along the axis of a cigar-shaped trap, one can use the estimate

$$ N \sim |\psi|^2 a_\perp^2 Z_0 $$

(see (3)) to write the corresponding condition as follows (e.g., see [11]):

$$ N \lambda_\perp / Z_0 \ll 1, \quad (7) $$

If this condition is satisfied, then the condensate wave function can be factorized:

$$ \psi(\mathbf{r}, t) = \phi(x, y) \Psi(z, t), \quad (8) $$

where

$$ \phi(x, y) = \frac{1}{\sqrt{\pi a_\perp}} \exp \left( \frac{x^2 + y^2}{2a_\perp^2} \right) \quad (9) $$

is the wave function of the ground state of transverse motion. Substituting (8) and (9) into (4) and (5) and integrating the result over the condensate’s cross section, one obtains the action expressed in terms of the one-dimensional Lagrangian density

$$ \mathcal{L}_{1D} = \frac{i \hbar}{2} (\Psi^* \dot{\Psi} - \dot{\Psi} \Psi^*) + \frac{\hbar^2}{2m} |\Psi'|^2 + \frac{1}{2} a_{\parallel}^2 |\Psi'|^2 + g_{1D} |\Psi'|^4, \quad (11) $$

where

$$ g_{1D} = \frac{g}{2\pi a_\perp^2} = \frac{2 \hbar^2 a_\parallel}{ma_\perp^2} \quad (12) $$

is an effective coupling constant and $\Psi$ is normalized as

$$ \int |\Psi|^2 dz = N. \quad (13) $$

Equation (11) determines the longitudinal dynamics of a condensate in a cigar-shaped trap.

By the well-known substitution

$$ \Psi(z, t) = \sqrt{\rho(z, t)} \exp \left( \frac{im}{\hbar} \int v(z', t) dz' \right), \quad (14) $$

Eq. (11) is transformed into the system

$$ \rho_z + \rho v_z + \frac{g_{1D}}{m} \rho + \omega_\parallel^2 \rho = 0, \quad (15) $$

$$ v_z + v v_z + \frac{g_{1D}}{m} v + \omega_\parallel^2 + \frac{\hbar^2}{2m} \rho \left( \frac{\rho_z^2}{4\rho^2} - \frac{\rho_z^2}{2\rho} \right) = 0. \quad (16) $$

In Eq. (16), the last term (“quantum pressure”) can be neglected if it is much smaller than the nonlinear term, i.e., if

$$ \frac{a_\parallel}{Z_0} \ll \frac{Na_\parallel}{a_\perp}. \quad (17) $$

Then, Eq. (16) reduces to

$$ v_z + v v_z + \frac{g_{1D}}{m} v + \omega_\parallel^2 z = 0. \quad (18) $$