On the Internal Nonlinear Resonant Three-Mode Interaction of Charged Drop Oscillations

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Abstract—The difference between internal nonlinear three-mode degenerate and Raman resonances is found for the first time: in the former case, the energy spent on the initial deformation of a drop is only transferred from lower to higher modes; in the latter case, it is transferred in both directions. It turns out that degenerate resonances are slightly sensitive to the physical quantities that are responsible for the exact positions of the resonances (i.e., to the amount of electric charge). A deviation from the resonant value only changes the fraction of the energy the modes exchange and the time of resonant energy exchange: the interaction itself remains resonant. © 2005 Pleiades Publishing, Inc.

(1) Among the effects related to nonlinear oscillations of a charged conducting incompressible liquid drop, internal nonlinear resonant interaction between oscillation modes occupies a prominent place, to judge by the number of publications concerned with this issue. From the pioneering works [1–5], appearing 20 years ago, to this day [6–18], more than three-fourths of the articles touching upon the problem of nonlinear oscillations of a charged drop have been devoted to the internal resonant interaction in one way or another. The fact is that resonant interaction provides the fastest and most efficient redistribution of the initial deformation energy of the drop between the modes excited via nonlinear interaction. To put otherwise, resonant interaction exerts a decisive effect both on the development of nonlinear oscillations (and, accordingly, the generation of their related acoustic and electromagnetic radiations [12, 14]) and on the disintegration of the drop that bears a near-critical (in terms of linear stability) charge [2, 5, 9, 11, 15, 17]. In spite of a great deal of articles devoted to resonant mode interaction, many related issues remain to be understood, the direction of resonant energy transfer between modes among them. Degenerate three-mode resonances, where one of two modes interacts with the other twice, were discovered and studied first [1–3]. It was argued [10, 16] that the energy in such resonances is transferred only from lower to higher modes, which, generally speaking, is inconsistent with the concept of disintegration instability as applied to three-mode interactions [19]. Furthermore, it was found [13] that disintegration instability may occur at truly three-mode resonances (secondary Raman resonances), specifically, that there are a number of resonant situations when the energy is transferred from two higher modes to a third (lower) one. However, the parameters of such an interaction (the time and amount of interaction) have not been explored. In [17], where four-mode interactions were studied, it was also demonstrated that energy may be transferred from higher to lower modes but the intensity of such transfer is low, since these interactions have the third order of smallness. Energy transfer from higher modes of nonlinear oscillations to lower ones (more specifically, to the fundamental mode) is of special interest in view of the mechanism of corona-initiated lightning discharge near a coarse highly charged drop, which is being discussed in the literature [15, 18].

In this work, we perform a detailed study of the energy transfer between modes that occurs in degenerate and secondary Raman resonances under three-mode interaction.

(2) Consider the time evolution of the surface of a nonlinearly oscillating drop of an ideal incompressible conducting liquid. The drop has a radius $R$, density $\rho$, surface tension coefficient $\gamma$, and charge $Q$ that is uniformly distributed over the surface. At the zero time $t = 0$, the equilibrium shape of the drop experiences an axisymmetric perturbation of fixed amplitude that is much smaller than the radius of the drop. Our aim is to find the spectrum of oscillations of the drop at $t > 0$.

We assume that the drop is axisymmetric from the initial time on, so that the equation of its surface in the spherical coordinate system with the origin placed at the center of the drop has the form

$$r(\theta, t) = 1 + \xi(\theta, t); \quad |\xi| \ll 1. \quad (1)$$

(we use dimensional variables such that $\rho = R = \gamma = 1$).

The flow of the liquid in the drop is assumed to be potential with a velocity field potential $\psi(r, t)$. Velocity field $V(r, t)$, in turn, is specified by a potential gradient: $V(r, t) = \nabla \psi(r, t)$. If the hydrodynamic velocity of the liquid in the drop is taken to be much lower than the
propagation velocity of electromagnetic interactions, the electric field of charge \( Q \) near the drop can be assumed to be electrostatic. Then, it can be described by potential \( \Phi(\mathbf{r}, t) \), which is related to field strength \( \mathbf{E} \) as \( \mathbf{E} = -\text{grad}(\Phi) \).

Mathematically, the problem is stated as

\[
\Delta \psi(\mathbf{r}, t) = 0; \quad \Delta \Phi(\mathbf{r}, t) = 0; \quad r \rightarrow 0: \psi(\mathbf{r}, t) \rightarrow 0; \quad r \rightarrow \infty: |\text{grad}(\Phi(\mathbf{r}, t))| \rightarrow 0; \quad r = 1 + \xi(\theta, t); \quad \frac{\partial \xi}{\partial t} = \frac{\partial \psi}{\partial r} - \frac{1}{r^2} \frac{\partial \psi}{\partial \theta};
\]

\[
\Delta \rho - \frac{\partial \psi}{\partial t} - \frac{1}{2}(\nabla \psi)^2 - \frac{1}{8\pi}(\nabla \Phi)^2 = \text{div} \mathbf{n};
\]

\[
\Phi(\mathbf{r}, \theta, t) = \text{const};
\]

\[
\int_V r^2 d\mathbf{r} \sin \theta d\theta d\phi = \frac{4}{3\pi},
\]

\[
V = [0 \leq r \leq 1 + \xi(\theta, t), 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi];
\]

\[
\int_V \mathbf{e}_r \cdot r^2 d\mathbf{r} \sin \theta d\theta d\phi = 0;
\]

\[
\frac{1}{4\pi} \int_S (\mathbf{n} \cdot \nabla \Phi) ds = Q,
\]

\[
S = [r = 1 + \xi(\theta, t), 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi];
\]

\[
t = 0: \xi(\theta, t) = \xi_0 P_0(\mu) + \xi_1 P_1(\mu) + \varepsilon \sum_{i \in \Xi} h_i P_i(\mu);
\]

\[
\sum_{i \in \Xi} h_i = 1; \quad \frac{\partial \xi(\theta, t)}{\partial r} = 0.
\]

Here, \( \Delta \) is the Laplacian.

Since conditions (8) and (9) must be fulfilled at any, including initial, time instant, they define (at \( t = 0 \)) the amplitudes of the zeroth and first modes in the expansion of the equilibrium (spherical) shape of the drop, \( \xi(\theta, t) \), in Legendre polynomials. This means that the amplitudes of both modes cannot be taken arbitrarily: they will depend on the initial deformation.

In expressions (6)–(11), \( \mu = \cos \theta \); \( \Delta \rho \) is the difference in the pressures inside and outside the drop in equilibrium; \( \mathbf{n} \) is the unit normal vector to surface (1); \( \varepsilon \) is the amplitude of a small initial perturbation of the surface (the small parameter of the problem); \( P_i(\mu) \) are the \( i \)th-order Legendre polynomials; \( h_i \) are the coefficients specifying the partial contribution of an \( i \)th vibrational mode to the total initial perturbation; \( \Xi \) is a set of the numbers of initially excited vibrational modes; and

\[
\xi_0 = -\varepsilon^2 \sum_{m=1}^{\infty} \frac{h_i^2}{(2i+1)} + O(\varepsilon^3);
\]

\[
\xi_1 = -\varepsilon^2 \sum_{i \in \Xi} \frac{9i h_{i-1} h_i}{(2i-1)(2i+1)} + O(\varepsilon^3)
\]

are the constants that are found from conditions (8) and (9) at the zero time (in (12), they are given accurate to the third order of smallness in \( \varepsilon \)).

(3) To find a solution to the problem stated, we will take advantage of the method of many scales (used to solve similar problems considered in [2, 5–7, 9–18]). Desired functions \( \xi(\theta, t), \psi(\mathbf{r}, t), \) and \( \Phi(\mathbf{r}, t) \) are represented as series in powers of small parameter \( \varepsilon \) and are assumed to be independent not merely of time \( t \) but of various \( \varepsilon \)-defined times \( T_m = \varepsilon^m t \):

\[
\xi(\theta, t) = \sum_{m=1}^{\infty} \varepsilon^m \xi^{(m)}(\theta, T_0, T_1, \ldots);
\]

\[
\psi(\mathbf{r}, t) = \sum_{m=1}^{\infty} \varepsilon^m \psi^{(m)}(\mathbf{r}, \theta, T_0, T_1, \ldots);
\]

\[
\Phi(\mathbf{r}, t) = \sum_{m=0}^{\infty} \varepsilon^m \Phi^{(m)}(\mathbf{r}, \theta, T_0, T_1, \ldots).
\]

We will restrict our analysis to a quadratic approximation and seek for dependences of the desired quantities on time scales \( T_0 \) and \( T_1 \).

Substituting expansions (13) into set (2)–(11) and equating the terms of the same power of \( \varepsilon \), we arrive at a set of boundary-value problems for functions \( \xi^{(m)}(\theta), \psi^{(m)}(\mathbf{r}), \) and \( \Phi^{(m)}(\mathbf{r}) \). Obviously, each of functions \( \psi^{(m)}(\mathbf{r}) \) and \( \Phi^{(m)}(\mathbf{r}) \) must satisfy linear equations (2).

In the zeroth order of smallness, we obtain expressions for the electrostatic potential near an equilibrium (spherical) drop with charge \( Q \), \( \Phi^{(0)} = Q/r \).

The first- and second-order solutions to Eqs. (2) that satisfy boundedness conditions (3) and (4) are written in the form

\[
\psi^{(m)}(\mathbf{r}, \theta, T_0, T_1) = \sum_{n=1}^{\infty} D_n^{(m)}(T_0, T_1) r^n P_n(\mu) \quad (m = 1, 2);
\]

\[
\Phi^{(m)}(\mathbf{r}, \theta, T_0, T_1) = \sum_{n=0}^{\infty} E_n^{(m)}(T_0, T_1) r^{-(n+1)} P_n(\mu).
\]

First- and second-order corrections to the equilibrium surface of the drop are also represented as expan-