A Kinetic Approach to Bose–Einstein Condensates: Self-Phase Modulation and Bogoliubov Oscillations

J. T. Mendonça, R. Bingham, and P. K. Shukla

1. INTRODUCTION

Presently, Bose–Einstein condensates (BECs) provide one of the most active and creative areas of research in physics [1, 2]. The dynamics of BECs are usually described by a nonlinear Schrödinger equation (known in this field as the Gross–Pitaevskii equation (GPE) [3, 4]), which determines the evolution of a collective wavefunction of ultracold atoms in BECs, evolving in the mean field self-potential.

In this paper, we propose the use of an alternative but nearly equivalent approach to the physics of BECs, based on a kinetic equation for the condensate. We also show that this kinetic theory can lead to a more complete understanding of the physical processes occurring in BECs, not only by providing an alternative method for describing the system but also by improving our global view of the physical phenomena. It is our hope that this will also lead to the discovery of new aspects of BECs.

The key point of our approach is the use of the Wigner–Moyal equation (WME) for BECs, describing the spatiotemporal evolution of the appropriate Wigner function [5]. Wigner functions for BECs were discussed in the past [6, 7], and the WME has been used sporadically [8], but no systematic application of the WME to BECs has previously been considered. In the semiclassical limit, this equation reduces to the particle-number conservation equation, which is a kinetic equation formally analogous to the Liouville equation, but with a nonlinear potential. A description of BECs in terms of the kinetic equation is adequate in a series of problems, as is exemplified here, and can be seen as intermediate (in accuracy) between the GPE and the hydrodynamic equations usually found in the literature.

This paper is organized as follows. In Section 2, we establish the WME and discuss its approximate version as a kinetic equation for the Wigner function. We then apply the kinetic equation to two distinct physical problems. The first one, considered in Section 3, is the self-phase modulation of a BE condensate beam, where we show that part of the beam is decelerated and eventually stops as a result of the gradient of the effective self-potential, and (ii) the derivation of a kinetic dispersion relation for sound waves in BECs, including collisionless Landau damping. © 2005 Pleiades Publishing, Inc.

2. WIGNER–MOYAL EQUATION FOR THE BOSE CONDENSATE

It is known that, for an ultracold atomic ensemble and, in particular, for BECs, the ground-state atomic quantum field can be replaced by a macroscopic atomic
wavefunction $\psi$. In a large variety of situations, the evolution of $\psi$ is determined by the GPE

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + (V_0 + V_{\text{eff}}) \psi, \quad (1)$$

where $V_0 \equiv V_0(r)$ is the confining potential and $V_{\text{eff}}$ is the effective potential that takes the interatomic interactions inside the condensate into account; in the simplest form,

$$V_{\text{eff}}(r, t) = g|\psi(r, t)|^2,$$

where $g$ is a constant [3, 4].

We consider the situation where this wave equation can be replaced by a kinetic equation. To construct such an equation, we introduce the Wigner function associated with $\psi$ via [5]

$$W(r, k, t) = \int \psi^*(r + \frac{s}{2}, t) \psi(r - \frac{s}{2}, t) \times \exp(-i k \cdot s) \, ds. \quad (2)$$

It is then possible to derive (see Appendix) the evolution equation for the Wigner function:

$$\left( \frac{\hbar^2}{2m} k \cdot \nabla - i\hbar \frac{\partial}{\partial t} \right) W = -2V(\sin \Lambda) W, \quad (3)$$

where

$$\Lambda = \left( \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial p} \right) \quad (4)$$

is a bidirectional differential operator that acts to the left on $V$ and to the right on $W$ [5]. In this equation, the potential is

$$V = V_0 + g \int W(r, k, t) \frac{d|k|}{(2\pi)^3} + \delta V, \quad (5)$$

where

$$\delta V = g \left( |\psi(r, t)|^2 - \int W(r, k, t) \frac{d|k|}{(2\pi)^3} \right) \quad (6)$$

can be considered a noise term associated with the square mean deviations of the quasiprobability, determined by the Wigner function $W$ with respect to the local quantum probability, which is determined by the wavefunction $\psi$.

Equation (3) can be seen as a WME describing the space and time evolution of BECs, and it is exactly equivalent to GPE (1). However, it is of little use in the above exact form, and it is convenient to introduce some simplifying assumptions. This is justified in the important case of slowly varying potentials. In this case, we can neglect the higher order spatial derivatives and introduce the approximation $\sin \Lambda \sim \Lambda$. This corresponds to the semiclassical approximation, where the quantum potential fluctuations can also be neglected, viz., $\delta V \rightarrow 0$. Introducing these two simplifying assumptions, valid in the semiclassical limit, we reduce the WME to the much simpler form

$$\left( \frac{\partial}{\partial t} + v \cdot \nabla + F \cdot \frac{\partial}{\partial k} \right) W = 0, \quad (7)$$

where $v = \hbar k/m$ is the velocity of the condensate atoms corresponding to the wavevector state $k$ and $F = -\nabla V$ is a force associated with the inhomogeneity of the condensate self-potential. The nonlinear term in GPE (1) is hidden inside this force $F$. As we see in what follows, this nonlinear term looks very much like a ponderomotive force term, similar to radiation pressure.

We note that this new equation is a closed kinetic equation for the Wigner function $W$. In this semiclassical limit, $W$ is just the particle occupation number for translational states with momentum $p = \hbar k$. Equation (7) is equivalent to a conservation equation, stating the conservation of the quasiprobability $W$ in the six-dimensional classical phase space $(r, k)$, and can also be written as

$$\frac{d}{dt} W(r, k, t) = 0. \quad (8)$$

This kinetic equation can then be used to describe physical processes occurring in a BEC, as long as the semiclassical approximation of slowly varying potentials is justified. The interest in such kinetic descriptions is illustrated with the aid of two simple and different examples, to be presented in the next two sections. Many other applications can be envisaged and will be explored in the future.

3. SELF-PHASE MODULATION
   OF A BEAM CONDENSATE

We first consider the kinetic description of self-phase modulation of a BEC gas moving with respect to the confining potential $V_0(r)$. Here, we can explore the similarity of this problem to that of self-phase modulation of short laser pulses moving in a nonlinear optical medium, which is well known in the literature [15]. To simplify our description, we consider the one-dimensional problem of a beam moving along the $z$ axis and neglect the axial variation of the background potential, $\partial V_0/\partial z = 0$. The radial structure of the beam can easily be introduced later and does not substantially modify...