SHORT COMMUNICATIONS

Asymptotic Behavior of Semiclassical Jointly-Orthogonal Polynomials of Bessel–Laguerre Type

R. É. Akhmedov*

Moscow State University

Received March 5, 2007

DOI: 10.1134/S0001434608030115

Key words: semiclassical joint orthogonality polynomial, Jacobi–Jacobi polynomial, Rodrigues formula, asymptotic behavior of orthogonal polynomials, saddle-point method.

The polynomials of degree \(2n\) defined by using the Rodrigues formula

\[
Q_{2n}(x) = e^{-\alpha/x} e^{-\beta x} \frac{d^n}{dx^n} [x^{2n} e^{\alpha/x} e^{\beta x}],
\]

are called the Bessel–Laguerre semiclassical joint orthogonality polynomials (see [1], [2]). The polynomials (1) satisfy the following orthogonality relations:

\[
\int_{\gamma_j} Q_{2n}(x)x^m w(x) \, dx = 0, \quad m = 0, \ldots, n - 1, \quad j = 1, 2,
\]

with the weight function

\[
w(x) = \exp\left\{\frac{\alpha}{x} + \beta x\right\}, \quad \alpha > 0, \quad \beta > 0,
\]
given on two contours in \(\mathbb{C}\): \(\gamma_1 = (-\infty, 0)\) and \(\gamma_2\) is a simple closed contour with the beginning and the end at the origin, and we have \(\text{Re}(\alpha/x) < 0, \quad x \in \gamma_2\), in a neighborhood of the origin.

The problem of the asymptotic behavior of joint orthogonality polynomials was considered in [3] (the Jacobi–Jacobi polynomials).

In the present paper, we study the asymptotic behavior of the scaled polynomials

\[
Q^*_{2n}(x) = Q_{2n}(nx), \quad (3a)
\]

\[
\tilde{Q}_{2n}(x) = Q_{2n}\left(\frac{x}{n}\right), \quad (3b)
\]

which gives a picture of the distribution of zeros. As follows from our results, \(n\) zeros of the polynomial \(Q^*_{2n}\) tend to zero as \(n \to \infty\), and the other \(n\) zeros fill an interval bounded away from zero; for the polynomials \(\tilde{Q}_{2n}\), \(n\) zeros “go” to infinity and the other zeros are distributed along a curve of Szegő type, exactly as is the case for the classical Bessel polynomials.

The formulas of strong asymptotics (Theorem 1 and Theorem 2) can be proved by using the saddle-point method [4]. The application of potential theory [5] makes it possible to find measures for the distribution of zeros (equilibrium measures) for the polynomials (3a) and (3b) (Corollaries 1 and 2).

To write out the asymptotic formulas, we need the functions

\[
p_\beta(t) = \log \frac{t-x}{t^2 e^{\beta(t-x)}}, \quad p_\alpha(t) = \log \frac{t-x}{t^2 e^{\alpha/t - \alpha/x}}.
\]

*E-mail: a-ruslan1@mail.ru.
Write
\[ t_j = t_j(x) = \frac{\beta x - 1}{2\beta} \pm \frac{1}{2\beta} \sqrt{\beta^2 x^2 + 6\beta x + 1}, \quad j = 1, 2, \] (5)
i.e., the \( t_j \) are the zeros of the derivative \( p'_\beta(t) \).

**Theorem 1.** Let
\[ \Delta := \left[ -\frac{3 - \sqrt{8}}{\beta}, -\frac{3 + \sqrt{8}}{\beta} \right], \quad \beta > 0. \]

For the polynomials (3a), the following asymptotic formulas hold:
\[ Q_{2n}^*(x) = \frac{1}{(2\beta)^n \sqrt{\pi n}} A_1(t_1)(\beta^2 x^2 + 4\beta x - 1 + (\beta x + 1)\sqrt{D})^n \]
\[ \times \exp\left\{ \left( \frac{\beta x - 1 + \sqrt{D}}{2} \right) n \right\} \left( 1 + O\left( \frac{1}{\sqrt{n}} \right) \right), \quad n \to \infty, \] (6)
uniformly with respect to \( x \) on compact subsets of the domain \( G = \mathbb{C} \setminus (\Delta \cup \{0\}) \);
\[ Q_{2n}^*(x) = \frac{1}{(2\beta)^n \sqrt{\pi n}} \left( A_1(t_1)(\beta^2 x^2 + 4\beta x - 1 + (\beta x + 1)\sqrt{D})^n \exp\left\{ \left( \frac{\beta x - 1 + \sqrt{D}}{2} \right) n \right\} \right. \]
\[ + A_1(t_2)(\beta^2 x^2 + 4\beta x - 1 - (\beta x + 1)\sqrt{D})^n \exp\left\{ \left( \frac{\beta x - 1 - \sqrt{D}}{2} \right) n \right\} \]
\[ \times \left( 1 + O\left( \frac{1}{\sqrt{n}} \right) \right), \quad n \to \infty, \] (7)
uniformly with respect to \( x \) on compact subsets of \( \Delta \). Here
\[ D = \beta^2 x^2 + 6\beta x + 1, \]
the points \( t_j \) are defined in (5), \( A_1(t) \) is a function analytic in a neighborhood of the point \( t_j \), and the principal branch of the function \( \sqrt{D} \) is chosen everywhere in formulas (6) and (7).

The principal term of the asymptotic expansion (6) and (7) enables one to find the measure of the distribution of zeros of the polynomials (3a), \( d\mu(x) \).

Let \( \{x_{j,n}\}_{j=1}^{2n} \) be the sequence of zeros of the polynomial \( Q_{2n}^*(x) \). Write
\[ Q_{n,1}(x) = \prod_{j=1}^{2n} \frac{x - x_{j,n}}{x_{j,n} - x_{j,n}}, \quad \nu_{n,1} = \frac{1}{n} \sum_{j=1}^{2n} \delta(x - x_{j,n}). \] (8)

**Corollary 1.** The sequence of measures \( \nu_{n,1} \) which count the zeros of the polynomials \( Q_{n,1}(x) \) in (8) weakly tends as \( n \to \infty \) to a measure \( \mu(x) \) having the density
\[ d\mu(x) = -\frac{1}{2\pi} \frac{\sqrt{(\beta x - \delta_+)(\delta_- - \beta x)}}{x} dx, \quad \delta_\pm = -3 \pm \sqrt{8}. \]
The measure \( \mu \) is equilibrium in the external field of a charge concentrated at the origin and the field \( -\text{Re}(\beta x) \). The potential of the measure \( \mu \) satisfies the equilibrium condition
\[ 2V_\mu(x) - \text{Re}(\beta x) + \ln \frac{1}{|x|} = w, \quad x \in \Delta. \]