SHORT

COMMUNICATIONS

On the Logarithmic Coefficients and Integral Means of Univalent Functions*

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Suppose $S$ is the class of functions

$$f(z) = z + a_2 z^2 + \cdots$$

analytic and univalent in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. The logarithmic coefficients $\gamma_n$ of $f(z)$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in D.$$

The Koebe function $k_\eta(z) = z(1 - \eta z)^{-2}$ has the logarithmic coefficients $\gamma_n = \eta^n/n$, where $|\eta| = 1$. It is known that the inequality $|\gamma_n| \leq 1/n$ holds for all spiral-like functions $f(z)$ in $S$, but is false for close-to-convex functions [1] and for the full class $S$, even in order of magnitude [2, Theorem 8.4]. Nevertheless, Milin has shown that, in a certain average sense, $\gamma_n$ cannot be much larger than $1/n$ (see [2] and [3]). The well-known Milin’s lemma gives

$$\sum_{k=1}^{n} k|\gamma_k|^2 \leq \sum_{k=1}^{n} \frac{1}{k} + \delta, \quad f \in S, \quad n = 1, 2, \ldots, \quad (1)$$

where

$$\delta = \frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{(\log 2)^n}{n! n} - \log \log 2 - c \right) = 0.3118 \ldots$$

and $c$ is the Euler constant. The Milin constant $\delta$ in (1) cannot be reduced to zero. The surprising proof of the Milin conjecture by de Branges [4] asserts that $\delta = 0$ in an average sense. That is,

$$M_n = M_n(f) := \frac{1}{n} \sum_{m=1}^{n-1} \sum_{k=1}^{m} \left( 1 - k|\gamma_k|^2 \right) \geq 0, \quad f \in S, \quad n = 2, 3, \ldots, \quad (2)$$

which implies the famous Bieberbach conjecture from the Lebedev–Milin inequality that $|a_n| \leq n e^{-M_n}$ (see [5], [6]). The sign of equality in (2) holds only for the Koebe function. One consequence of (2) is

$$\sum_{k=1}^{\infty} k|\gamma_k|^2 r^k \leq \sum_{k=1}^{\infty} \frac{1}{k} r^k = \log \frac{1}{1-r}, \quad f \in S, \quad 0 \leq r < 1 \quad (3)$$

(see [7]).

*The text was submitted by the author in English.

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It is well known that, for $0 \leq r < 1$, the integral means of the logarithmic derivative of $f \in S$ is

$$I_2^2(r, \frac{zf'(z)}{f(z)}) := \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{r f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta = 1 + 4 \sum_{k=1}^{\infty} k^2 |\gamma_k|^2 r^{2k}. \quad (4)$$

By applying (3) with $r$ on the left-hand side replaced by $r^2$ and the inequality

$$kr^k \leq \frac{e^{-1}}{1 - r},$$

Kayumov [8] obtained the following inequality

$$I_2^2(r, \frac{zf'(z)}{f(z)}) \leq 1 + \frac{4}{e} \frac{\log(1 - r)}{1 - r}, \quad f \in S, \quad 0 \leq r < 1, \quad (5)$$

which can be used to prove Brennan’s conjecture for a special class of functions. In [8, p. 500], Kayumov proposed the following question.

**Question.** Is it possible to replace the “logarithmic” multiplier in (5) by an absolute constant?

For the case in which the Hayman index $\alpha(f)$ of $f \in S$ satisfies $\alpha(f) > 0$, Kayumov [8] gave a positive answer to this question.

In the present paper, we will simply show that the answer to this question is negative.

We first note that $\sqrt{r}$ in (5) can be replaced by $r$ in a simple way. We then answer the above question negatively. In fact, if the "logarithmic" multiplier in (5) can be replaced by an absolute constant, then

$$I_2^2(r, \frac{zf'(z)}{f(z)}) = O\left(\frac{1}{1 - r}\right) \quad \text{as} \quad r \to 1, \quad (6)$$

which is equivalent to

$$\sum_{k=1}^{n} k^2 |\gamma_k|^2 = O(n) \quad \text{as} \quad n \to \infty \quad (7)$$

for $f \in S$ (see [9, Lemma 1]). Hence we have from (7),

$$\sum_{k=1}^{n} k^2 |\gamma_k|^2 = o(n \log n) \quad \text{as} \quad n \to \infty, \quad (8)$$

which is equivalent to

$$I_2^2(r, \frac{zf'(z)}{f(z)}) = o\left(\frac{1}{1 - r \log \frac{1}{1 - r}}\right) \quad \text{as} \quad r \to 1 \quad (9)$$

for $f \in S$. However, Hayman [10] has constructed a function $f \in S$ for which (9) does not hold (see also [2, p. 272]). Thus, it is not true that one can replace the “logarithmic” multiplier in (5) by an absolute constant.

Note that (8) or (9) holds for $f \in S$ with $o$ replaced by $O$, while (6) or (7) holds for each function $f \in S$ whose image has finite area (see [9]). Hence one can also give a positive answer to the above question for each function $f \in S$ whose image has finite area or whose logarithmic coefficients satisfy $|\gamma_n| \leq c/n$ for all $n \geq 1$ and an absolute constant $c > 0$.

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