SHORT COMMUNICATIONS

On the Number of Domains of Maximal Dimension in Partitions of Projective Spaces by Hyperplane Arrangements

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1. INTRODUCTION

Grünenbaum [1] systematically studied the combinatorial characteristics of sets of $n$ lines on the projective plane; these characteristics include the possible values of and relations between the numbers $v$ of points of intersection, $e$ of line segments, $f$ of domains, $t_i$ of points of intersection of multiplicity $i$, and $p_j$ of $j$-angle domains. In particular, he partly described the set of possible values of the number $f$ of connected components (domains) in the complement of a union of lines (not necessarily in general position). Martinov [2] completely described the range of $f$ for sets of lines and pseudo-lines. Shannon [3] obtained sharp lower bounds for the numbers of $k$-dimensional planes and $k$-dimensional cells in partitions of the real projective space $\mathbb{RP}^d$ by nontrivial sets of $n$ hyperplanes and found sets on which the bounds are attained. Zaslavsky [4] expressed the number of surfaces in the complement of a hyperplane arrangement in terms of the intersection complex of the hyperplanes. Buck [5] found the maximum possible number of $k$-dimensional cells in partitions of $\mathbb{RP}^d$ by sets of $n$ hyperplanes.

The present paper continues and generalizes the cited results due to Grünenbaum, Shannon, and Martinov; we use Zaslavsky’s results and properties of geometric lattices. Our results may have applications owing to the relationships between the combinatorics of sets $\mathcal{A}$ of real hyperplanes and the topology of the complements $\mathbb{C}^n \setminus \mathcal{A}$ to the complexified hyperplanes. For example, the number $f$ of domains in the real arrangement coincides with the dimension of the cohomology ring of the complement $\mathbb{C}^n \setminus \mathcal{A}$; see Orlic–Solomon [6]. Deshpande [7] generalized the characteristic polynomial to arrangements of submanifolds, and so the present paper can be generalized to arrangements of submanifolds as well with appropriate inequalities (bounds). There are also results on (hyper)plane arrangements due to numerous other authors, which we do not mention here because we do not use them directly. The paper might have a relationship with the estimate of the number of threshold functions in binary logic. The number of threshold functions (Boolean $n$-ary functions specified by the condition $\sum a_i x_i \geq c$) coincides with the number of domains in the specific configuration formed by the $2^n$ hyperplanes $\pm a_1 \pm a_2 \cdots \pm a_n = c$. Thus, the lower bounds obtained in the paper for the number of domains in degenerate hyperplane arrangements can be used to improve lower bounds for the number of threshold functions.

The statement of the problem is as follows. Consider a union of $n$ distinct hyperplanes of dimension $d - 1$ in the real $d$-dimensional projective space $\mathbb{RP}^d$, where the intersection of these $n$ hyperplanes is empty. Describe the range $F_d(n)$ of the number $f$ of connected components in the complement of this union in $\mathbb{RP}^d$. Arrange the elements of $F_d(n)$ in ascending order, find the first few elements, and prove that the integers between them cannot be realized as the number of domains.

The following theorem is the main result of the present paper.

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Theorem. Let \( d \geq 3 \) and \( n \geq 2d + 5 \). Then the first four elements of the set \( F_d(n) \) in ascending order are as follows:

\[
(n - d + 1)2^{d-1}, \quad 3(n - d)2^{d-2}, \quad (3n - 3d + 1)2^{d-2}, \quad 7(n - d)2^{d-3}.
\]

Unlike Shannon, we seek not only (one) minimal number of domains, but the first few (in ascending order) possible numbers of domains. (The first one of them naturally coincides with the Shannon number.) In contrast to Martinov [2] and Shnurnikov [8], here we consider hyperplane arrangements in \( \mathbb{RP}^d \) for \( d \geq 3 \). For \( d = 3 \), the first three numbers in \( F_d(n) \) given by the theorem do not exceed \( 6n - 16 \) and coincide with the numbers found in [8].

2. DEFINITIONS AND AUXILIARY RESULTS

Two projective subspaces of dimensions \( i \) and \( j \) in \( \mathbb{RP}^d \) are said to be in general position if their intersection is an \( (i + j - d) \)-dimensional projective space for \( i + j \geq d \) and an empty set for \( i + j < d \). We say that a projective subspace is in general position with respect to a hyperplane arrangement if it is in general position with each of the hyperplanes and each of their nonempty intersections. A hyperplane arrangement is trivial if all the hyperplanes have a common point. A near pencil is a (nontrivial) arrangement of \( n \) hyperplanes in \( \mathbb{RP}^d \) in which \( n - d + 1 \) hyperplanes have a common \( (d - 2) \)-dimensional plane and each of the other \( d - 1 \) hyperplanes is in general position with respect to the remaining \( n - 1 \) hyperplanes. For an arrangement \( \mathcal{A} \) of \( n \) hyperplanes of dimension \( d - 1 \) in \( \mathbb{RP}^d \), let \( f = f(\mathcal{A}) \) be the number of connected components of the complement in \( \mathbb{RP}^d \) to the union of the hyperplanes forming the arrangement; the connected components themselves will be referred to as domains. The set of all possible \( f \) for nontrivial hyperplane arrangements will be denoted by \( F_d(n) \).

The maximal element of \( F_d(n) \) is

\[
1 + \binom{n - 1}{1} + \cdots + \binom{n - 1}{d}
\]

for \( d \leq n - 1 \); it is attained on hyperplane arrangements in general position (see [5]). If \( \mathcal{A} \) is a hyperplane arrangement in \( \mathbb{RP}^d \), then \( m = m(\mathcal{A}) \) stands for the maximum number of hyperplanes having a point in common.

Lemma 1 (Shannon [3]). The minimal element of \( F_d(n) \) is \((n - d + 1)2^{d-1}\), and it is only realized by near pencils.

Let \( p \) be one of the hyperplanes in an arrangement \( \mathcal{A} \). By \( \mathcal{A}' \) we denote the arrangement formed by all hyperplanes in \( \mathcal{A} \) except for \( p \). Next, we denote by \( \mathcal{A}'' \) the arrangement formed by the intersections of \( p \) with the hyperplanes in \( \mathcal{A}' \); i.e., \( \mathcal{A}'' \) is an arrangement of \( (d - 2) \)-dimensional planes in the \( (d - 1) \)-dimensional plane \( p \). The triple \((\mathcal{A}, \mathcal{A}', \mathcal{A}'')\) will be called a triple of arrangements. Let \( L(\mathcal{A}) \) be the set of all nonempty intersections (including the entire projective space) of hyperplanes in \( \mathcal{A} \) ordered by reverse inclusion. The elements of \( L(\mathcal{A}) \) will be called planes.

Definition. The M"obius function \( \mu: L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z} \) is the function defined recursively as follows:

\[
\mu(u, v) = \begin{cases} 
0 & \text{if } u \not\in v; \\
1 & \text{if } u = v; \\
- \sum_{u \leq w < v} \mu(u, w) & \text{if } u < v.
\end{cases}
\]

If the first argument of the M"obius function is the ambient space \( V \), then we omit it; i.e., we write \( \mu(w) \) rather than \( \mu(V, w) \) for a plane \( w \). For the sets of intersections of the arrangements \( \mathcal{A}, \mathcal{A}', \) and \( \mathcal{A}'' \), we denote the respective M"obius functions by \( \mu, \mu', \) and \( \mu'' \).

Lemma 2 (Zaslavsky [4]). For a triple \((\mathcal{A}, \mathcal{A}', \mathcal{A}'')\) of arrangements,

\[
f(\mathcal{A}) = f(\mathcal{A}') + f(\mathcal{A}'').
\]