DETERMINATE SYSTEMS

The Investigation Algorithm of Stability of Periodic Oscillations in the Problem for the Andronov–Hopf Bifurcation

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Received July 2, 2008

Abstract—A new method for analysis of the stability of free oscillations under the conditions of the Andronov–Hopf bifurcation is suggested. In contrast to commonly applied methods, the algorithm suggested does not require the construction of integral varieties and the transition to normal forms. The algorithm is based on the comparison between the characters of stability of the stationary state of the system and the free oscillations being born. The method suggested enables us to simplify essentially the analysis of stability and obtain simple explicit criteria of stability and instability, and also define the type of bifurcation.

PACS number: 05.45.-a

DOI: 10.1134/S0005117908120035

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

A dynamic system is considered, the operation of which in the space of states is described by the equation

\[ x' = A(\lambda)x + a(x, \lambda), \quad x \in \mathbb{R}^N, \quad N \geq 2, \quad \lambda \in \mathbb{R}^1. \tag{1} \]

Here, the matrix \(A(\lambda)\) and the vector function \(a(x, \lambda)\) are continuous in \(x\) and \(\lambda\), with

\[ \lim_{|x| \to 0} \max_{|\lambda - \lambda_0| \leq 1} \frac{|a(x, \lambda)|}{|x|} = 0. \tag{2} \]

The sign \(|\cdot|\) is used both for the designation of absolute values of numbers and for Euclidean norms of vectors and matrices. It is assumed that the following condition is fulfilled:

(U1) the matrix \(A(\lambda_0)\) has simple eigenvalues \(\pm \omega_0 i\), \(\omega_0 > 0\), in which case its all eigenvalues different from \(\pm \omega_0 i\) have negative real parts.

In view of (2), the Eq. (1) at all \(\lambda \in [\lambda_0 - 1, \lambda_0 + 1]\) has the zero solution. In the transition of the parameter \(\lambda\) through the value \(\lambda_0\), the parameter in the neighborhood of the stationary state \(x = 0\) can display the nonstationary periodic solutions of a small amplitude with a period close to \(T_0 = 2\pi/\omega_0\).

The pair \((\lambda_0, T_0)\) is called the [1, 2] point of the Andronov–Hopf bifurcation of the system (1) if to each \(\varepsilon > 0\) there corresponds such a value \(\lambda = \lambda_\varepsilon \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)\), as that at which the system (1) has a nonzero \(T_\varepsilon\)-periodic solution \(x_\varepsilon(t)\), where \(T_\varepsilon \in (T_0 - \varepsilon, T_0 + \varepsilon)\) and \(|x_\varepsilon(t)| \leq \varepsilon, -\infty < t < \infty\). A certain cycle corresponds to the solution \(x_\varepsilon(t)\), while values \(\lambda_\varepsilon\) fill, as a rule,
the interval of the form \((\lambda_0 - \varepsilon_0, \lambda_0)\) or \((\lambda_0, \lambda_0 + \varepsilon_0)\), i.e., the bifurcating solutions \(x_\varepsilon(t)\) generally exist only at \(\lambda < \lambda_0\), or only at \(\lambda > \lambda_0\). In the first case, we will speak of the subcritical bifurcation, while in the second case, of the supercritical bifurcation.

We will present, in the convenient form for us, the basic assertion (see, for example, [1,2]) as to the Andronov–Hopf bifurcation under the following conditions:

(U2) The matrix \(A(\lambda)\) and the vector function \(a(x, \lambda)\) are continuously differentiable with respect to \(\lambda\), in which case

\[
a(x, \lambda) = a_2(x, \lambda) + a_3(x, \lambda) + \varepsilon(x, \lambda),
\]

where \(a_2(x, \lambda)\) and \(a_3(x, \lambda)\) are terms of the order of 2 and 3 in \(x\), \(\varepsilon(x, \lambda) = o(|x|^3)\) and \(|\varepsilon(x, \lambda) - \varepsilon(y, \lambda)| \leq \beta(r)|x - y|\) for \(|x|, |y| \leq r\); here \(\beta(r) = o(r^2)\).

In view of the condition (U1) at small \(|\lambda - \lambda_0|\), the matrix \(A(\lambda)\) has simple eigenvalues \(\mu(\lambda) = \alpha(\lambda) \pm i\omega(\lambda)\), where the functions \(\alpha(\lambda)\) and \(\omega(\lambda)\) are continuous, in which case \(\alpha(\lambda_0) = 0\) and \(\omega(\lambda_0) = \omega_0\).

(U3) The relation \(\alpha'(\lambda_0) \neq 0\) is valid.

**Theorem 1.** Let the conditions (U1)–(U3) be fulfilled. Then the pair \((\lambda_0, T_0)\) will be the Andronov–Hopf bifurcation point for the (1). In this case, the periodic solutions being born exist at small \(|\lambda - \lambda_0|\) exactly in one of the three cases: (1) \(\lambda > \lambda_0\); (2) \(\lambda < \lambda_0\); (3) \(\lambda = \lambda_0\); here, in the first two cases, no more than one cycle of a small amplitude corresponds to each \(\lambda\).

**Remark 1.** The Theorem 1 remains valid also in the case when the second part of the condition (U1) is replaced by a weaker supposition: the matrix \(A(\lambda_0)\) has no eigenvalues of the form \(\pm k\omega_0i\), \(k = 0, 2, 3, \ldots\).

One of the basic questions in the study of the Andronov–Hopf bifurcation is the question of stability of arising cycles and also the question related to it of the type of bifurcation (subcritical or supercritical). We note that if the matrix \(A(\lambda_0)\) has at least one eigenvalue with the positive real part, then the periodic solutions being born of the system will certainly be unstable.

The questions as to stability of the arising cycles and the type of bifurcation are essentially more complex than the problem for features of the Andronov–Hopf bifurcation of the system (1), which is commonly defined (as in the Theorem 1) only by the properties of the matrix \(A(\lambda)\). But the determination of the type of bifurcation and the character of stability of bifurcating solutions requires the accounting of nonlinear summands up to the third order inclusively. The works of many authors are devoted to the investigation of the stated questions (see, for example [1–5]). The algorithms suggested are generally complex and call for calculations by cumbersome formulas.

Here, one of the basic approaches is the following (see, for example, [1,2]). If the system (1) has the dimension that is more than two \((N \geq 3)\), then it is previously projected on the central manifold of the system, which must be known, at least to quadratic terms. The obtained two-dimensional system (or, at \(N = 2\), the initial system) is brought to the normal form by the special replacement of variables, the coefficients of which define the character of stability of bifurcating solutions of the system (1). The solution of the stated problems generally involves complex calculations.

In [5], another algorithm is suggested for the investigation of stability, which uses the pattern of M. Rozo [6] and the method of functioning of a parameter, which is suggested by M.A. Krasnosel’skii [7]. This algorithm is also complex, and its realization is a substantial action.

In this work, the new diagram is suggested for the investigation of stability of the cycles being born, which leads to substantially simpler calculations than the above-enumerated algorithms. The diagram is based on the effective use of the fact that in the natural statement between the cycles