DETERMINATE SYSTEMS

Linear Systems with Multiple Singular Values

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Received March 14, 2008

Abstract—Properties of linear dynamical systems are studied in the case of Hankel singular values of high multiplicity for monosingular and bisingular systems. Singular values of the Hankel operator take only one or two values for such systems. Properties of transfer functions are analysed. Algorithms of transfer function analysis and design are developed. Relations between frequency characteristics and Hankel singular values are established.

PACS number: 02.30.Yy

DOI: 10.1134/S0005117909010044

1. INTRODUCTION

The Hankel operator and its singular values are of great importance for modern theory of linear stationary systems [1, 2]. They are used to solve approximation and reduction problems for linear systems, to develop optimal consistent models, to design robust control systems, etc. [3–8]. The Hankel singular values appear in a natural way while constructing a balanced representation. They form diagonal entries of the controllability and observability Gramians [5, 9, 10].

Most papers consider systems with different Hankel singular values. However, it is also important for theoretical and engineering problems to consider the case of multiple singular values. This very case is considered in the present paper. We pay a special attention to the case of maximum multiplicity, where either all singular values are the same, or they are divided into two groups of same values. Such systems are called monosingular or bisingular, respectively [11, 12].

Different descriptions are used in the linear control system analysis and design. The most common among them are: matrix description (state space method); operator description (transfer functions); structure description (flowcharts and signal graphs); frequency characteristics (amplitude-frequency characteristics, phase-frequency characteristics, amplitude-phase characteristics). When the Hankel operator and its singular values are studied, the matrix description is mainly used, while other descriptions are not so common. In particular, there is no definition of Hankel singular values in terms of transfer functions or frequency characteristics; canonical forms and structure realizations of transfer functions which correspond to the balanced representation are unknown; design problems for systems with given Hankel singular values are not solved in terms of transfer functions.

In this paper we solve the above problems for monosingular and bisingular systems.

The Hankel singular values represent input-output characteristics of the system. Since they do not depend on a particular basis in state space, they should be closely connected with descriptions such as transfer functions and frequency characteristics. This connection is especially evident in the case of monosingular and bisingular systems. We show in the paper that the Hankel singular values of such systems determine “overall dimensions” of the Nyquist diagram and can be found directly from transfer functions or frequency characteristics, without computing Gramians.

1 This work was supported by the Russian Foundation for Basic Research, project nos. 08-08-00228.
The structure of the paper is as follows:

In Section 2 we present necessary information about Hankel singular values and balanced representations of linear SISO systems.

Section 3 is devoted to monosingular systems, i.e., systems with identical singular values. There is a well-known example of such systems: the all-pass systems. We obtain criteria of monosingularity a structure of balanced representation for transfer functions.

In Section 4 we define bisingular systems and describe two types of their decomposition into monosingular subsystems. We analyse properties of bisingular systems and present algorithms to find Hankel singular values directly from transfer functions.

In Section 5 we state and solve the design problem for bisingular transfer functions with given Hankel singular values.

Each Section is illustrated by examples.

2. THE HANKEL SINGULAR VALUES AND BALANCED REPRESENTATION

Let’s give basic information about Hankel singular values. Consider a stable linear stationary system of order $n$ with single input and single output (SISO) described in state space as

$$
\dot{X} = AX + bu, \quad y = cX + du,
$$

(2.1)

where $A$ is a square $n \times n$ matrix; $b$ and $c$ are column vector and row vector, respectively; $d$ is a constant; $u$ and $y$ are input and output, $X$ is the state vector.

The transfer function of the system is $Q(p) = B(p)/A(p)$, where $A(p)$, $B(p)$ are polynomials of degree not greater than $n$.

Among the important input-output characteristics of system (2.1) there are Hankel singular values, in addition to transfer function zeros and poles. The conventional way to find them is based on the analysis of the controllability and observability Gramians $W_c$ and $W_o$. The latters satisfy the matrix Lyapunov equations

$$
AW_c + W_c A^T + bb^T = 0, \quad A^T W_o + W_o A + c^T c = 0.
$$

(2.2)

The eigenvalues of the product $W_c W_o$ don’t depend on a particular basis in state space. If the system is stable, controllable, and observable, all those eigenvalues are real and positive.

**Definition 2.1.** The positive number $\sigma_1, \ldots, \sigma_n$ being the arithmetic square roots of the eigenvalues of the product $W_c W_o$ are called the (arithmetic) Hankel singular values of system (2.1).

The numbers $\sigma_i$ represent the gain properties of the system, the maximum $\sigma_i$ being equal to the Hankel norm of the transfer function. The numbers are input-output invariants of the system.

By a linear change of variables system (2.1) can be reduced to that having identical diagonal Gramians:

$$
W_c = W_o = \text{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0.
$$

(2.3)

The Gramians’ diagonal entries are composed of Hankel singular values.

**Definition 2.2.** The realization of system (2.1) that satisfy (2.3) is called the balanced representation. For scalar systems we can always ensure that all entries of the column vector $b$ are non-negative.