SCHEDULING PROBLEMS ON A SINGLE MACHINE

Minimizing Total Weighted Completion Time with Uncertain Data: A Stability Approach

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Abstract—A single-machine scheduling problem is investigated provided that the input data are uncertain: The processing time of a job can take any real value from the given segment. The criterion is to minimize the total weighted completion time for the \( n \) jobs. As a solution concept to such a scheduling problem with an uncertain input data, it is reasonable to consider a minimal dominant set of job permutations containing an optimal permutation for each possible realization of the job processing times. To find an optimal or approximate permutation to be realized, we look for a permutation with the largest stability box being a subset of the stability region. We develop a branch-and-bound algorithm to construct a permutation with the largest volume of a stability box. If several permutations have the same volume of a stability box, we select one of them due to one of two simple heuristics. The efficiency of the constructed permutations (how close they are to a factually optimal permutation) and the efficiency of the developed software (average CPU-time used for an instance) are demonstrated on a wide set of randomly generated instances with \( 5 \leq n \leq 100 \).

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1. INTRODUCTION

Real-life scheduling problems may involve different forms of uncertainty and several approaches are available in the OR literature to deal with uncertain scheduling problems. In the well-developed stochastic approach [1], an uncertain processing time is assumed to be a random variable with a probability distribution which is known a priori. If there is no sufficient information to characterize a priori the probability distribution of each random processing time, other approaches are needed [2, 3]. In particular, in a robust approach [4–6], the decision-maker prefers a schedule that hedges against the worst-case scenario among all the possible realizations of the uncertain processing times. The stability approach developed in [7–12] combines a stability analysis, a two-stage scheduling decision framework, and the solution concept of a minimal dominant set of schedules. A minimal dominant set of schedules being constructed in an off-line fashion optimally covers all the possible scenarios: For any possible scenario such a set contains at least one schedule which is optimal [9–11]. A minimal dominant set of schedules (which may be constructed off-line) allows a scheduler to make an on-line decision whenever additional information on the realization of the processing times becomes available [9, 10].

In this paper, we consider a single-machine scheduling problem with interval processing times of \( n \) jobs which have to be processed. To solve this problem optimally (or approximately), we use the notion of a stability box of a job permutation, which is similar to the well-known stability ball [9, 12–14] used in post-optimality analysis. We use an exact formula for characterizing the

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stability box of any concrete job permutation in $O(n \log n)$ time and develop a fast branch-and-bound algorithm to find a permutation with the largest volume of a stability box. We report computational results on finding a permutation with the largest volume of a stability box and satisfying one of two heuristic rules.

2. PROBLEM SETTING, NOTATIONS, STATE-OF-THE-ART

A set of $n$ jobs $\mathcal{J} = \{J_{1}, J_{2}, \ldots, J_{n}\}$, $n \geq 2$, has to be processed on a single machine, a weight $w_{i} > 0$ being given for job $J_{i} \in \mathcal{J}$. The processing time $p_{i}$ of a job $J_{i} \in \mathcal{J}$ can take any real value between a lower bound $p_{i}^{L} > 0$ and an upper bound $p_{i}^{U} \geq p_{i}^{L}$, both bounds being known before scheduling. The processing time $p_{i}$ may remain unknown until the completion of job $J_{i}$ (such a condition is realistic for most real-life scheduling problems).

Let $T$ denote the set of all vectors $p = (p_{1}, p_{2}, \ldots, p_{n})$ of the possible processing times. The set $T$ is a closed rectangular box in the space $R_{+}^{n}$ of non-negative $n$-dimensional real vectors and may be represented as the Cartesian product of the $n$ segments $[p_{i}^{L}, p_{i}^{U}]$, $i \in \{1, 2, \ldots, n\}$:

$$T = \{ p \mid p \in R_{+}^{n}, \ p_{i}^{L} \leq p_{i} \leq p_{i}^{U}, \ i \in \{1, 2, \ldots, n\} \} = \times_{i=1}^{n} [p_{i}^{L}, p_{i}^{U}].$$

A vector $p \in T$ of the possible processing times is called a scenario.

Let $S = \{\pi_{1}, \pi_{2}, \ldots, \pi_{n!}\}$ be the set of all permutations $\pi_{k} = (J_{k_{1}}, J_{k_{2}}, \ldots, J_{k_{n}})$ of the jobs $\mathcal{J} = \{J_{1}, J_{2}, \ldots, J_{n}\}$. If both the permutation $\pi_{k}$ of the job set $\mathcal{J}$ and the scenario $p \in T$ are fixed, then $C_{i} = C_{i}(\pi_{k}, p)$ is the completion time of job $J_{i} \in \mathcal{J}$ in a semi-active schedule defined by permutation $\pi_{k}$. As usual, a schedule is called semi-active if no job $J_{i} \in \mathcal{J}$ can start earlier without delaying the completion time of another job from set $\mathcal{J}$ and without altering the processing permutation of the jobs $\mathcal{J}$. The criterion under consideration is $\sum w_{i}C_{i}$, which is the minimization of the sum of the weighted job completion times:

$$\sum_{J_{i} \in \mathcal{J}} w_{i}C_{i}(\pi_{i}, p) = \min_{\pi_{k} \in S} \{ \sum_{J_{i} \in \mathcal{J}} w_{i}C_{i}(\pi_{k}, p) \},$$

where permutation $\pi_{i} = (J_{i_{1}}, J_{i_{2}}, \ldots, J_{i_{n}}) \in S$ is optimal. By adopting the three-field notation $\alpha|\beta|\gamma$ introduced in [15], the above scheduling problem is denoted by $1|p_{i}^{L} \leq p_{i} \leq p_{i}^{U}|\sum w_{i}C_{i}$.

Since the scenario $p \in T$ may remain unknown before the completion of the jobs $\mathcal{J}$, the completion time $C_{i}$ of each job $J_{i} \in \mathcal{J}$ cannot be calculated at the stage of scheduling. Therefore, problem $1|p_{i}^{L} \leq p_{i} \leq p_{i}^{U}|\sum w_{i}C_{i}$ is not mathematically correct in the sense that the values of the objective function $\gamma = \sum_{J_{i} \in \mathcal{J}} w_{i}C_{i}(\pi_{k}, p)$ for different permutations $\pi_{k} \in S$ remain uncertain before the completion of the job set $\mathcal{J}$.

If a scenario $p \in T$ is fixed before scheduling (i.e., equalities $p_{i}^{L} = p_{i}^{U} = p_{i}$ hold and segment $[p_{i}^{L}, p_{i}^{U}]$ is degenerated into one point $p_{i} = [p_{i}, p_{i}]$ for each job $J_{i}$, $i \in \{1, 2, \ldots, n\}$), then problem $1|p_{i}^{L} \leq p_{i} \leq p_{i}^{U}|\sum w_{i}C_{i}$ reduces to problem $1||\sum w_{i}C_{i}$ with deterministic processing times, which is mathematically correct and can be solved in $O(n \log n)$ time due to Smith [16].

In what follows, the problem $1|p_{i}^{L} \leq p_{i} \leq p_{i}^{U}|\gamma$ with the objective function $\gamma = f(C_{1}, C_{2}, \ldots, C_{n})$ is called uncertain in contrast to its counterpart, problem $1||\gamma$, which is called deterministic. While an optimal sequencing rule for the deterministic problem $1||\sum w_{i}C_{i}$ has been known since 1956 [16], its uncertain counterpart $1|p_{i}^{L} \leq p_{i} \leq p_{i}^{U}|\sum w_{i}C_{i}$ continues to attract the attention of the researchers who develop different approaches for correcting and solving the uncertain problem $1|p_{i}^{L} \leq p_{i} \leq p_{i}^{U}|\sum w_{i}C_{i}$ (see [4–7, 11, 17, 18] among others). Next, we survey some recent results for scheduling problems with uncertain processing times.